Geometric singular perturbation analysis of a Autocatalator model

Peter Szmolyan

Vienna University of Technology

joint work with: Ilona Kosiuk
Overview

- 2-d Autocatalator, slow-fast structure
- relaxation oscillation, asymptotics, numerics
- a rescaling
- two scaling Regimes
- blow-up analysis
Autocatalator Model

\[ \begin{align*}
\dot{a} &= \mu - a - ab^2 \\
\varepsilon \dot{b} &= -b + a + ab^2
\end{align*} \] (1)

\( a \) slow, \( b \) fast, \( 0 < \varepsilon \ll 1 \), parameter \( \mu > 0 \)

Fast time scale \( \tau := t/\varepsilon \)

\[ \begin{align*}
a' &= \varepsilon(\mu - a - ab^2) \\
b' &= -b + a + ab^2
\end{align*} \] (2)
Slow-fast subsystems

The limiting systems for $\varepsilon = 0$:

- **the reduced problem**
  \[
  \dot{a} = \mu - a - ab^2 \\
  0 = -b + a + ab^2
  \]  
  \(3\)

- **the layer problem**
  \[
  a' = 0 \\
  b' = -b + a + ab^2
  \]  
  \(4\)
Dynamics of the layer problem

critical manifold

\[ S = \{ a - b + ab^2 = 0 \} \]

equilibria of

layer problem

\[ S \text{ graph} \]

\[ a = \frac{b}{b^2 + 1}, \ b \geq 0 \]
Fast dynamics

$\mathcal{S}$ has

- attracting branch $\mathcal{S}_a$, $b < 1$
- repelling branch $\mathcal{S}_r$, $b > 1$
- non-hyperbolic fold point $p_f = (\frac{1}{2}, 1)$
Dynamics of the reduced problem

Differentiate $a = \frac{b}{1+b^2}$ with respect $t$

$$\dot{a} = \frac{1 - b^2}{(1 + b^2)^2} \dot{b} = \mu - b$$

- singular at $b = 1$, unless $\mu = 1$ (canard!)
- equilibrium $b = \mu$
- $\dot{a} > 0$, $b < \mu$, $\dot{a} < 0$, $b > \mu$
Slow dynamics

Three different cases:

\[ \mu < 1, \text{excitable} \]
Slow dynamics

Three different cases:

$\mu < 1$, excitable

$\mu = 1$, canard

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Slow dynamics

Three different cases:

\[ \mu < 1, \text{excitable} \]

\[ \mu = 1, \text{canard} \]

\[ \mu > 1, \text{jump point} \]
Slow dynamics

Three different cases:

- \( \mu < 1 \), excitable
- \( \mu = 1 \), canard
- \( \mu > 1 \), jump point

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For $\mu > 1$ we have a jump point at fold

good control for $\varepsilon \ll 1$, attraction onto $S_{a,\varepsilon}$ followed by jump, map: $\pi : \Sigma_{in} \to \Sigma_{out}$
Should we expect relaxation oscillation?
Should we expect relaxation oscillation?

Let’s ask the computer!
A rather “big” surprise

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Computers also scale things!

rescale by zooming in!
This is closer to what we just proved
What did go wrong?

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Let’s check again what we did

\[ a' = \varepsilon (\mu - a - ab^2) \]

\[ b' = -b + a + ab^2 \]
Let’s check again what we did

\[ a' = \varepsilon (\mu - a - ab^2) \]
\[ b' = -b + a + ab^2 \]

*a* slow, only valid for *a*, *ab^2* bounded!!!
Let’s check again what we did

\[ a' = \varepsilon(\mu - a - ab^2) \]
\[ b' = -b + a + ab^2 \]

\( a \) slow, only valid for \( a, ab^2 \) bounded!!!

\( a \) bounded, but \( b \) gets very large!
Rescaling for large $b$

GSPT valid for $b = O(1)$

$$a' = \varepsilon(\mu - a - ab^2)$$

$$b' = -b + a + ab^2$$

- $b = O(1/\varepsilon) \Rightarrow$ new scales, different asymptotic analysis
- critical manifold not compact $\Rightarrow$ loss of normal hyperbolicity
Rescaling gives a super-fast system

New variables

\[ a = A, \quad b = \frac{B}{\varepsilon}, \quad T = t/\varepsilon^2 \]

Rescaled system

\[ A' = \mu \varepsilon^2 - A\varepsilon^2 - AB^2 \]
\[ B' = -B\varepsilon + A\varepsilon^2 + AB^2 \]

where \( \prime \) denotes differentiation with respect to \( T \).
The rescaled layer problem is simple but degenerate

\[ \varepsilon = 0 \text{ in rescaled system} \]

\[ A' = -AB^2 \]
\[ B' = AB^2 \]

Critical manifold: two lines of equilibria
- \( A = 0 \) normally hyperbolic, attracting
- \( B = 0 \) non-hyperbolic, weakly repelling
∃ very degenerate singular periodic orbit $\gamma_0$

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Critical manifold $A = 0$ is normally hyperbolic

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Critical manifold $A = 0$ perturbs to slow manifold

$M_0 = \{(0, B) : B \in [\beta_0, \beta_1], \beta_0 > 0\}$ normally hyperbolic

**Theorem:** $\exists$ attracting slow manifold $M_\varepsilon$, given as

$$A = h(B, \varepsilon), \quad B \in [\beta_0, \beta_1]$$

with

$$h(B, \varepsilon) = \varepsilon^2 \frac{\mu}{B^2} + O(\varepsilon^3) \quad \text{singular as } B \to 0$$

slow flow on $M_\varepsilon$:

$$\frac{dB}{d\tau} = -B + O(\varepsilon), \quad \tau = \varepsilon T = t/\varepsilon$$
Return mechanism: reduced flow $B' = -B$
How did the folded critical manifold disappear?

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There exist two Regimes with good asymptotic analysis

Regime 1: $b = O(1)$  
Regime 2: $B = O(1), b = 1/\varepsilon$
The fold is hidden in the nonhyperbolic line $B = 0$!!!
Can this be combined to prove existence of relaxation oscillation?

Regime 1: \( b = O(1) \)  
Regime 2: \( B = O(1), b = 1/\varepsilon \)
Overlap, matching, proof: ????????

Regime 1: $b = O(1)$   Regime 2: $B = O(1), b = 1/\varepsilon$
Overlap, matching, proof: blow-up

Regime 1: \( b = O(1) \)  \hspace{1cm}  Regime 2: \( B = O(1), b = 1/\varepsilon \)
Main theorem

Theorem

For $\mu > 1$ and all $\varepsilon$ sufficiently small there exists a unique periodic orbit $\gamma_\varepsilon$ of system (5) and hence of the equivalent system (1) which tends to the singular cycle $\gamma_0$ for $\varepsilon \to 0$. 
Proof: Blow-up analysis based on the extended system

\[ A' = \mu \varepsilon^2 - A\varepsilon^2 - AB^2 \]
\[ B' = -\varepsilon B + A\varepsilon^2 + AB^2 \]
\[ \varepsilon' = 0 \]  \hspace{1cm} (6)

- degenerate line \( l_A \) of equilibria: \( B = 0, \varepsilon = 0 \)
- linearization at \((A, 0, 0)\): triple eigenvalue \( \lambda = 0 \)

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The degenerate line is blown-up to a cylinder

blow up transformation

\[ A = \bar{a} \]
\[ B = r \bar{b} \]
\[ \varepsilon = r \bar{\varepsilon} \] (7)

with

\[ \bar{a} \in \mathbb{R}, \quad \bar{b}^2 + \bar{\varepsilon}^2 = 1, \quad r \in \mathbb{R} \]

line \( l_A \) blown up to cylinder \( \mathbb{R} \times S^1 \), i.e. \( r = 0 \)
Blow-up transformation and charts $K_1$ and $K_2$
Blow-up transformation and dynamics is described in charts

- chart $K_1$: $\bar{\epsilon} = 1$, scaling chart
  \[ A = a_1, \quad B = r_1 b_1, \quad \epsilon = r_1 \]

- chart $K_2$: $\bar{b} = 1$, "compactification"
  \[ A = a_2, \quad B = r_2, \quad \epsilon = r_2 \epsilon_2 \]
**Chart** $K_1$ covers Regime 1, i.e. the $(a, b, \varepsilon)$ system

Equations in chart $K_1$:

\[
\begin{align*}
a'_1 &= r_1(\mu - a_1 - a_1 b_1^2) \\
b'_1 &= -b_1 + a_1 + a_1 b_1^2 \\
r'_1 &= 0
\end{align*}
\]

This is the original system with

\[
a = a_1, \quad b = b_1, \quad \varepsilon = r_1,
\]

transforming to chart $K_1 \Leftrightarrow$ undoing rescaling
Chart $K_2$ covers Regime 2 and overlaps with Regime 1

equations in chart $K_2$

\begin{align*}
a' &= -r(a + \varepsilon^2a - \varepsilon^2\mu) \\
r' &= r(a + \varepsilon^2a - \varepsilon) \\
\varepsilon' &= -\varepsilon(a + \varepsilon^2a - \varepsilon)
\end{align*} \tag{8}

here ' denotes differentiation with respect to a rescaled time variable $t_2$. 

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In chart $K_2$ we meet old friends!

invariant subspaces

- $\varepsilon = 0$
  \[
  a' = -ar \\
  r' = ar
  \]

critical manifolds

$L_s, L_b, S$

- $r = 0$
  \[
  a' = 0 \\
  \varepsilon' = (\varepsilon - a - \varepsilon^2 a)\varepsilon
  \]

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In the blown-up space there exists a desingularized singular cycle $\Gamma_0 = \omega_1 \cup \omega_2 \cup \omega_3 \cup \omega_4 \cup \omega_5$
Existence of the limit cycle

**Theorem**

For sufficiently small $\varepsilon$ the blown up vector field $\bar{X}$ has a family of periodic orbits $\bar{\Gamma}_\varepsilon$ which for $\varepsilon = r\bar{\varepsilon} = 0$ tend to the singular cycle

$$\Gamma_0 = \omega_1 \cup \omega_2 \cup \omega_3 \cup \omega_4 \cup \omega_5.$$
Proof is based on Poincare map $\Pi : \Sigma_1 \rightarrow \Sigma_1$

$\Pi_1 : \Sigma_1 \rightarrow \Sigma_2$ – passage of the fold point $p_f$
$\Pi_2 : \Sigma_2 \rightarrow \Sigma_3$ – passage of the hyperbolic line $L_s$
$\Pi_3 : \Sigma_3 \rightarrow \Sigma_4$ – contraction and slow drift toward the vertical slow manifold
$\Pi_4 : \Sigma_4 \rightarrow \Sigma_5$ – passage of the nilpotent point $q$
$\Pi_5 : \Sigma_5 \rightarrow \Sigma_1$ – transition towards the attracting slow manifold.

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Proof uses (known) blow-up of fold-point $p_f$ and needs new (simple) blow-up of point $q$
Why did it work?

- desingularization by blow up
- hidden details are visible
- blow-up gives scaling and overlap
- perturb from a well behaved limiting object
- gain of hyperbolicity
- GSPT becomes applicable