Continuation in Slow-Fast systems

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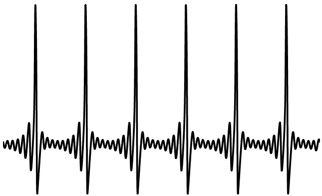
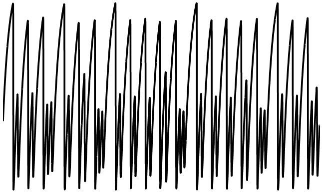
Oscillations with different modes . . .

. . . “Mixed-Mode Oscillations” (MMOs)



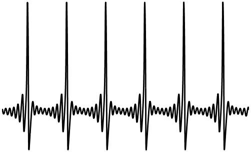


Oscillations with different modes . . .

. . . “Mixed-Mode Oscillations” (MMOs)

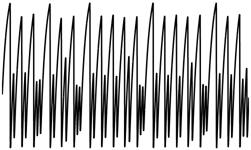
① What type of dynamical system to model MMOs?

⇒ **slow-fast dynamical systems**



② What numerical tool?

⇒ **numerical continuation**



Outline

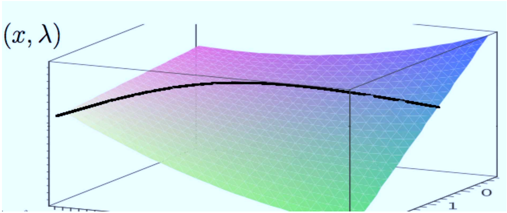
Numerical continuation : idea

- Goal is to compute families (branches) of solutions of nonlinear equations of the form:

$$\boxed{\mathrm{F}(\mathrm{x}) = 0,\ \mathrm{F} : \mathbb{R}^{n+1} \to \mathbb{R}^{n}}$$

➠ under-determined system (one more unknowns than equations)

➠ away from singularities, solution set = 1-dim. manifold embedded in (n+1)-dim. space



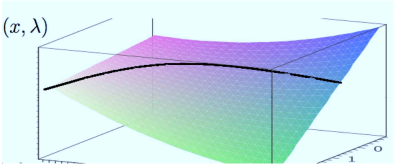
Numerical continuation : idea

- Many problems can be put in this form $\boxed{\mathrm{F}(\mathrm{x}) = 0,\ \mathrm{F} : \mathbb{R}^{n+1} \to \mathbb{R}^n}$

- In particular, discretisation of parameterised ODEs: $\boxed{\dot{x} = \mathrm{F}(x, \lambda)}$

  ➠ stationary problems (search for equilibria)

  ➠ Boundary Value Problems (BVP), including periodic orbits



This will rely on the application of the Implicit Function Theorem !

Parameter continuation

- Suppose we have one solution to the problem and wish to vary one component to find a new solution ...

$$\mathrm{F}(\mathrm{x}_0) = 0,\ \mathrm{x}_0 = (x_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$$

- In the I.F.T. holds at this point, then locally there is a branch of solutions parameterised by λ: (x(λ), λ).

- Small change in the parameter λ ⇒ new point that is **not** a solution of the problem but close to one!

$$\begin{aligned} \mathrm{x}_1^{\#} &= (x_0, \lambda_0 + \delta s), \\ \mathrm{F}(\mathrm{x}_1^{\#}) &\neq 0, \\ \mathrm{F}(\mathrm{x}_1^{\#}) &\ll 1 \end{aligned}$$

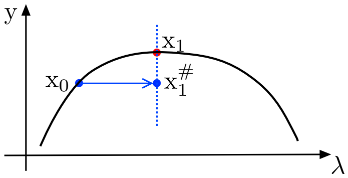
Parameter continuation

- **SO** initial guess for the new solution is $\mathrm{x}_1^{\#} = (x_0, \lambda_0 + \delta s)$

- New solution computed to a desired accuracy by using Newton's method on the augmented problem

$$\begin{aligned} \mathrm{F}(\mathrm{x}) &= 0, \\ \lambda - (\lambda_0 + \delta\lambda) &= 0 \end{aligned}$$

- **Note:** additional equation is to ensure unique solution for Newton's method

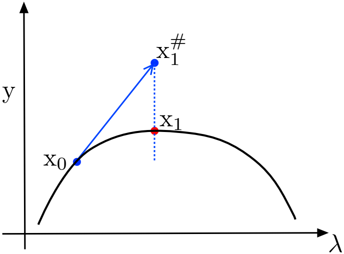


Tangent continuation

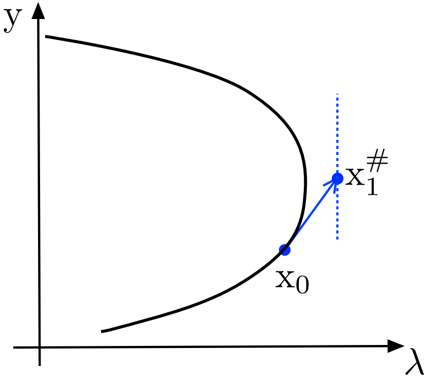
- **Improvement of the method:** use higher order initial guess x_1^* such as the tangent to the curve

- New solution computed with Newton's method on the same augmented problem

$$\begin{aligned} \mathrm{F}(\mathrm{x}) &= 0, \\ \lambda - (\lambda_0 + \delta\lambda) &= 0 \end{aligned}$$



Problem at a fold !!



Keller's pseudo-arclength continuation

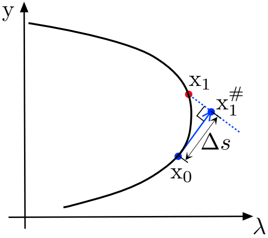
- **Problem at a fold:** the "parameter" chosen to do the continuation cannot parameterise the solution curve
- **Solution:** parameterise by something that do not have this problem

➠Arclength s along the curve !

- The problem to be solved becomes

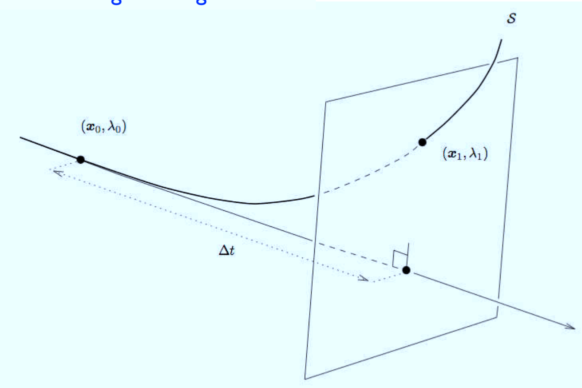
$$\begin{aligned} F(x) &= 0, \\ (\lambda - \lambda_0)\dot{\lambda}_0 + (x - x_0)\dot{x}_0 - \delta s &= 0 \end{aligned}$$

Arclength measured along the tangent space !



Keller's pseudo-arclength continuation

In practice: one mesures the projection of abscisse along the tangent!



Periodic orbit continuation

- We look for periodic solutions of the problem : $$\dot{x} = \mathrm{F}(x(t), \lambda)$$
- Fix the interval of periodicity by the transformation t → t/T such that : $$x' = T\mathrm{F}(x(t), \lambda)$$

  ➠ We seek for solutions of period 1 i.e. such that : $$x(1) = x(0)$$

  ➠ The true period T is now an additional parameter

- **Note:** the above equations do **not** uniquely specify x and T ...
  ... translation invariance !

➠ Necessity of a Phase condition

- <u>Example</u>: Poincaré orthogonality condition

$$(x_k(0) - x_{k-1}(0))^* x'_{k-1}(0)) = 0$$

- <u>In practice</u>: Integral phase condition

$$\int_0^1 x_k(t)^* x'_{k-1} dt = 0$$

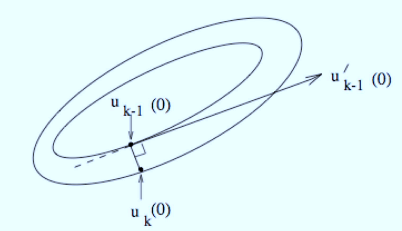
Periodic orbit continuation

- We look for periodic solutions of the problem : $\dot{x} = \mathrm{F}(x(t), \lambda)$
- Fix the interval of T

$' = T\mathrm{F}(x(t), \lambda)$

➠ We seek for $(1) = x(0)$

➠ The true peri



- **Note:** the above T ...

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Periodic orbit continuation

- ... we then solve a large G(X) = 0 augmented by the arclength-continuation equation:

$$\int_0^1 (x_k(t) - x_{k-1}(t))^* \dot{x}_{k-1} dt + (T_k - T_{k-1})\dot{T}_{k-1}$$
$$+(\lambda_k - \lambda_{k-1})\dot{\lambda}_{k-1} - \delta s = 0$$

- **Discretisation of the periodic orbits:** using orthogonal collocation (piecewise polynomials on mesh intervals)

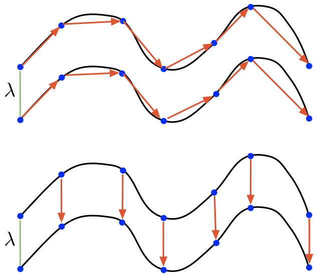
  ➠ Solve exactly at mesh points (boundaries of mesh intervals)

  ➠ Inside mesh intervals: well-chosen collocation points (good convergence properties)

- **Well-posedness:** n+1 unknowns (n components of x and period T) for n+1 conditions (periodicity + phase)

  ➠ varying 1 parameter will give a 1-parameter family of per. orbits

Better handling of the error



➠ ‘Boundary-Value Problem (BVP)’ vs. ‘Initial-Value Problem (IVP)’ : control at both ends, error ‘spread’ along the orbit instead of being concentrated at one end (shooting)

Continuation of
Boundary-Value Problems (BVPs)

- Solve G(X) = 0 augmented by the pseudo-arclength equation

- **Difference:** more generales boundary conditions B(F(a),F(b))=0,
  and integral conditions I(F(a),F(b))=0

- Same discretisation of the orbit segment i.e. by collocation

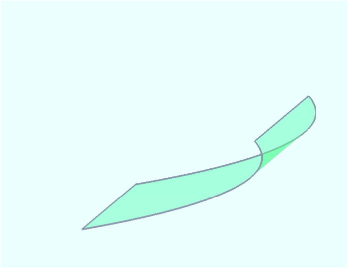
- **Problem is well posed:**

  # Bound. cond. + # int. cond. - dim. + 1 = # free parameters

- **Application:** numerical computation of piece of 2D invariant manifolds ...

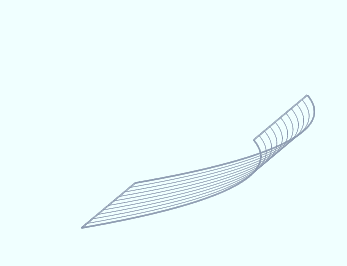
Computing 2D manifolds

- a surface

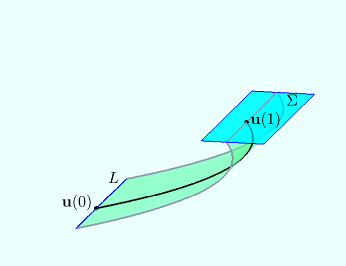


Computing 2D manifolds

- see as a one-parameter family of orbit segments



Computing 2D manifolds



- <u>each orbit segment</u>: solution to a boundary-value problem of the form

$$\boxed{\begin{array}{lcl} \dot{\mathbf{u}} & = & T\mathbf{g}(\mathbf{u},\lambda) \\ \mathbf{u}(0) & \in & L \\ \mathbf{u}(1) & \in & \Sigma \end{array}}$$

$\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$

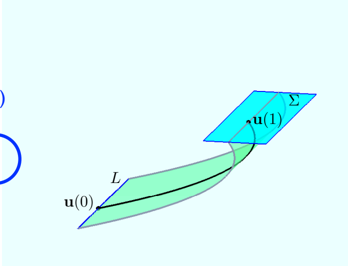
$T \in \mathbb{R},\ \lambda \in \mathbb{R}^p$

L, Σ: submanifolds of $\mathbb{R}^n$

Computing 2D manifolds

Family of well-posed BVPs: (isolated solution)

#Bound Cond. +1 = #Eq. + #Free Par.



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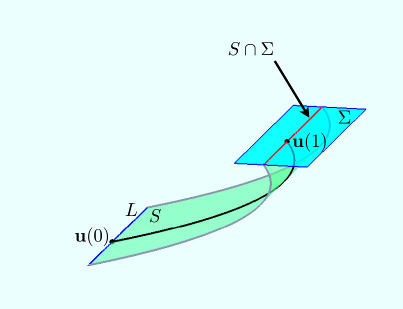
$\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$

$T \in \mathbb{R},\ \lambda \in \mathbb{R}^p$

$L,\ \Sigma$: submanifolds of $\mathbb{R}^n$

Method

- each orbit segment computed by collocation
- family of BVPs solved by numerical continuation
- this allows to compute a piece of interest of the manifold S



- note that the end point u(1) traces out the intersection curve $S \cap \Sigma$

Continuation vs. shooting

- strong convergence or divergence of trajectories towards one another

  $\Longrightarrow$ problem initial mesh

  $\Longrightarrow$ non-uniform covering of the manifold of interest

- numerical continuation well suited to slow-fast dynamical systems

  $\Longrightarrow$ extreme sensitivity to initial conditions

  $\Longrightarrow$ fast exponential instability of non-attracting slow manifolds

$\Longrightarrow$ shooting methods can fail!

Van der Pol with constant forcing a

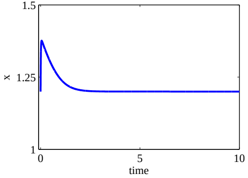
- 2nd order ODE: $\boxed{x_{tt} + \alpha(x^2 - 1)x_t + x = a}$
- Recast as a set of 1st order ODEs:

$$\varepsilon x_t = (y - \frac{x^3}{3} + x)$$
$$y_t = a - x$$

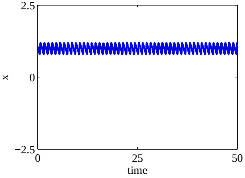
avec $\boxed{0 < \varepsilon = \frac{1}{\alpha} \ll 1}$.

- Dynamics as a is varied:

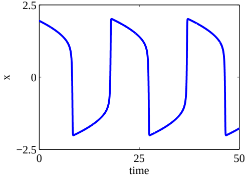
$a = 1.2$



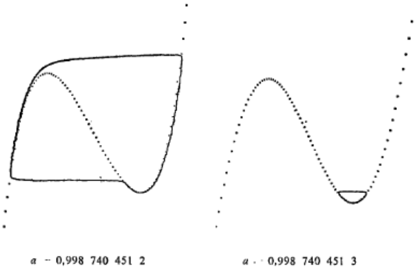
$a = 0.9935$



$a = 0.9934$



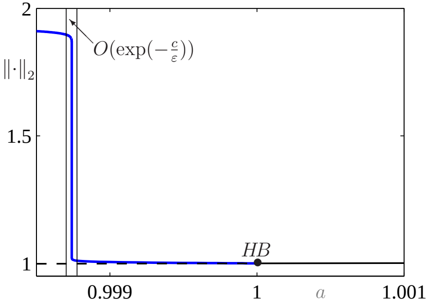
Benoît, Callot, Diener & Diener (1981)



How to interpret this figure?

- use time scales
- consider the singular limit $\varepsilon = 0$

Bifurcations when varying a



- Hopf bifurcation at $a = 1$
- the branch initially behaves as expected in $\sqrt{a}$ but ...

...at a distance of order $O(\varepsilon)$ from the Hopf point, the branch increases dramatically and becomes quasi-vertical!

Time scale analysis: $\varepsilon > 0 \Rightarrow \varepsilon = 0$

$x_t \sim O(1/\varepsilon) \implies x$ is **fast**	$y_t \sim O(1) \implies y$ is **slow**

- Limiting problem for the **slow** dynamics:

$\varepsilon > 0$

$$\varepsilon x_t = (y - \frac{1}{3}x^3 + x)$$
$$y_t = a - x$$

$\varepsilon = 0$: reduced system

$$0 = (y - \frac{1}{3}x^3 + x)$$
$$y_t = a - x$$

- Limiting problem for the **fast** dynamics:

$$\boxed{\tau = t/\varepsilon}$$

$\varepsilon > 0$

$$x_\tau = (y - \frac{1}{3}x^3 + x)$$
$$y_\tau = \varepsilon(a - x)$$

$\varepsilon = 0$: Layer system

$$x_\tau = (y - \frac{1}{3}x^3 + x)$$
$$y_\tau = 0$$

Time scale analysis: $\varepsilon > 0 \Rightarrow \varepsilon = 0$

$x_t \sim O(1/\varepsilon) \implies x$ is **fast**	$y_t \sim O(1) \implies y$ is **slow**

- Limiting problem for the **slow** dynamics:

$\varepsilon = 0$: reduced system

$$0 = (y - \frac{1}{3}x^3 + x)$$
$$y_t = a - x$$

slow system:

ODE defined on

$$S := \{y = \frac{1}{3}x^3 - x\}$$

- Limiting problem for the **fast** dynamics:

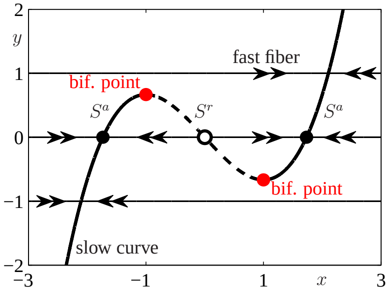
$\varepsilon = 0$: layer system

$$x_\tau = (y - \frac{1}{3}x^3 + x)$$
$$y_\tau = 0$$

fast system: family of ODEs parametrised by y

S = set of equilibria

Time scale analysis: $\varepsilon = 0 \Rightarrow \varepsilon > 0$

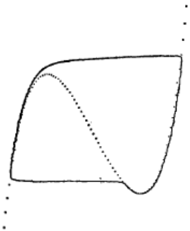


- outside a neighbourhood of the cubic S, the **fast** dynamics dominates
- in an ε-neighbourhood of S, the **slow** dynamics dominates
- transition: bifurcation points of **fast** dynamics

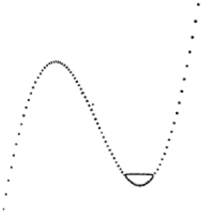
$\boxed{\textbf{Note}}$ S has 2 fold points $\Rightarrow$ stability is different on each side:

S^a **is attracting** and S^r **is repelling**

Back to Benoît et al.



$a = 0{,}998\ 740\ 451\ 2$



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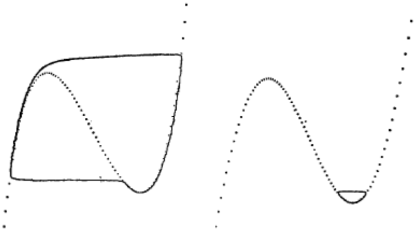
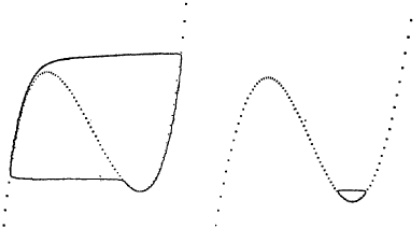
Back to Benoît et al.



$a = 0{,}998\ 740\ 451\ 2$ $a = 0{,}998\ 740\ 451\ 3$

- The Van der Pol system possesses unexpected limit cycles which
  - follow the attracting part S^a of the cubic S ...
  - down to the fold, and then ...
  - **follow the repelling part S^r of S!**
- these cycles have been named **canards** by the French mathematicians who discovered them.

Back to Benoît et al.



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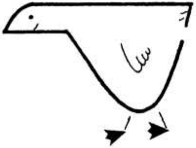
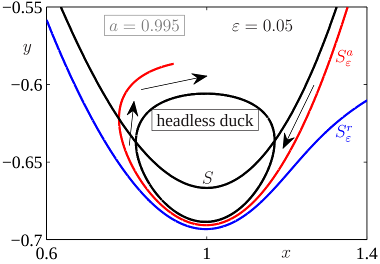
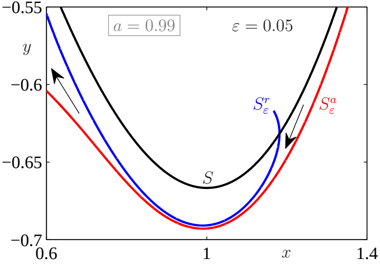


Invariant manifolds: $\varepsilon = 0 \Rightarrow \varepsilon > 0$

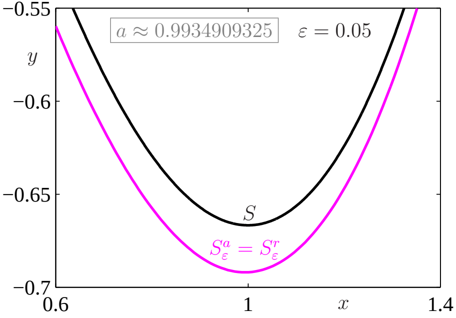
Fenichel Theory away from fold points, the attracting and repelling sheets S^a and S^r of S persist for $\varepsilon > 0$ (suff. small) as (locally) invariant manifolds S^a_ε and S^r_ε called slow manifolds.

Transition if a is decreased from the Hopf bifurcation point, $\Rightarrow S^a_\varepsilon$ et S^r_ε exchange their relative positions.



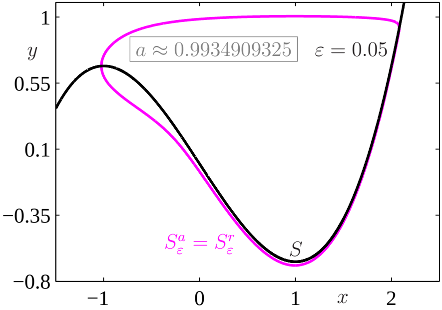


The canard point: when $S_\varepsilon^r = S_\varepsilon^a$



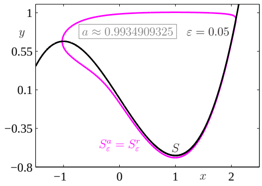
- the slow manifolds S_ε^a and S_ε^r pass through each other for $a = a_c(\varepsilon)$ called the point de canard
- the value $a_c(\varepsilon)$ can be obtained using an asympt. expansion in ε from the equation $S_\varepsilon^a = S_\varepsilon^r$, which defines the maximal canard

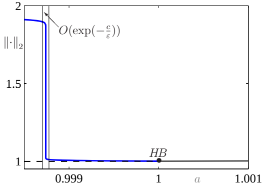
The canard point: when $S^r_\varepsilon = S^a_\varepsilon$

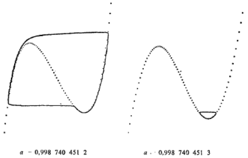


- the maximal canard follows S_r for the longest time, i.e., until the left fold point.

The canard point: when $S^r_\varepsilon = S^a_\varepsilon$

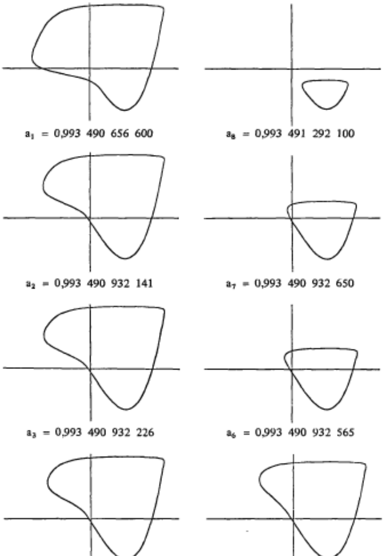






- the maximal canard:
  - transition between small-amplitude headless canards and large-amplitude canards with head
  - located in the upper part of the quasi-vertical segment of the branch of limit cycles
- this transition:
  - interval of a-values which is exponentially small en ε
  - extremely brutal (yet continuous!) event termed **canard explosion** by Brøns [*Math. Eng. Ind.* **2**: 51–63, 1988]

Easy to find canards numerically?



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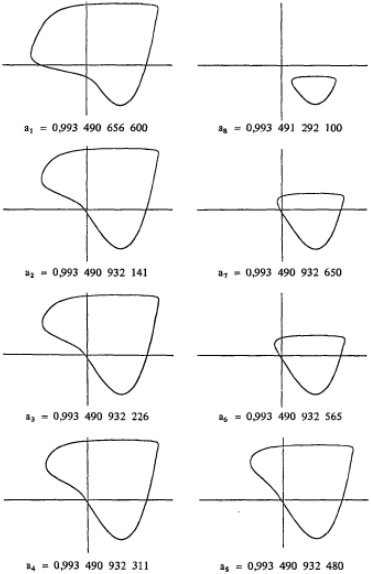


by direct simulations: **Delicate ...**

① see the values of a on the left!

$\Rightarrow$ such a system is extremely sensitive to initial conditions and to parameter variations . . .

② and here ε is “only” 0.05 . . . !

Already remarkable that this canard explosion could be computed by Benoît et al. at the end of the 1970s!!

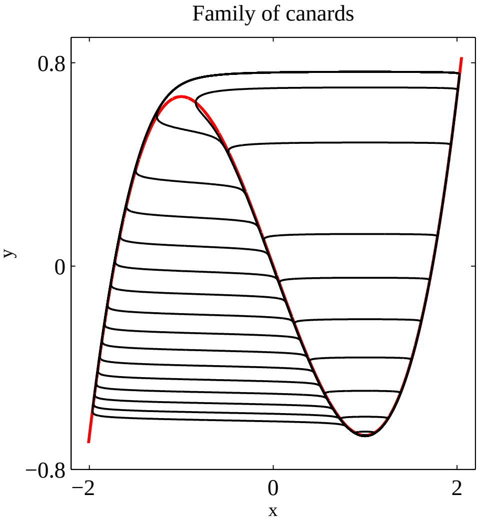
Problems when simulating slow-fast systems

- **extreme sensitivity to initial conditions** $\Rightarrow$ if ε is too small, one needs time steps getting closer to machine-precision
- direct simulation by shooting can be less reliable if the vector field has a **strong repulsion** in the normal direction
- lack of precision of initial value solvers can lead to spurious solution (fake chaos!)

One solution: **numerical continuation of periodic orbits**

- continuation: predictor-corrector method (Implicit Function Theorem)
- collocation: better handling of the error and good convergence properties
- integration time T becomes an unknown

Canard explosion computed by continuation



$\varepsilon = 0.005$

Example: the FitzHugh-Nagumo system

$$\boxed{\begin{aligned} v_t &= v - v^3/3 - w + I \\ w_t &= \varepsilon(v + a + bw) \end{aligned}}$$

Valeurs des Paramètres

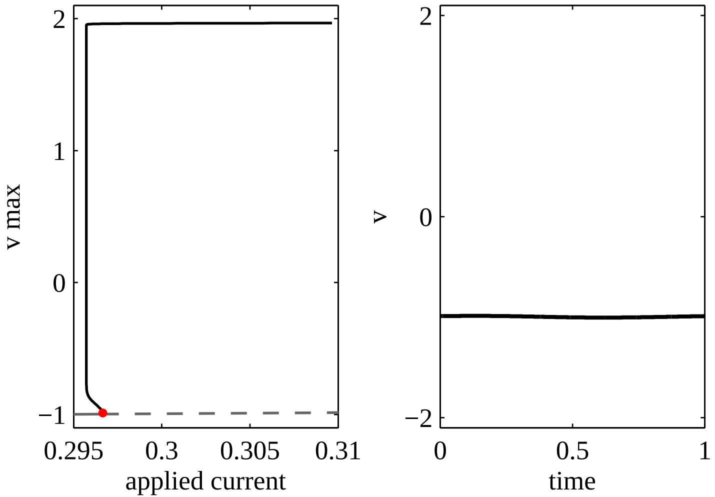
$\varepsilon = 0.01$

$a = 0.7$

$b = -0.8$

I : **bifurcation parameter**

Canard explosion → action potential



Summary for the 2D case: canard explosion

- interesting phenomenon, brutal but **non**-discontinuous!
- first discovered by direct simulation $\sim$ 30 years ago
- brutal because: exp. small (in ε) parameter variation
- much more general than Van der Pol!
- transition to the maximal canard only requires a non degenerate quadratic fold in the fast nullcline S $\Rightarrow$ robust
- encountered in **many** applications since the early 80s:
  - chemistry (Belousov-Zhabotinskii)
  - neuroscience (FitzHugh-Nagumo)
  - mechanical systems, electronic circuits, ...

$\boxed{\text{What if we add an extra slow variable ?}}$

Simplest way : add a constant slow drift

$$\varepsilon x_t = y - x^2$$
$$y_t = a - x$$
$$z_t = \mu$$

- x and y are **slow**
- z is **fast**

fast dynamics is trivial; let's focus on the **slow** one!

$\varepsilon = \mathbf{0}$: **reduced system**

$$0 = y - x^2$$
$$y_t = a - x$$
$$z_t = \mu$$

slow system

curve $\rightsquigarrow$ surface

$S := \{y = x^2\}$

called critical manifold

S: **parabola in $\mathbb{R}^2$ $\Longrightarrow$ parabolic cylinder in $\mathbb{R}^3$**

fold F: isolated points in $\mathbb{R}^2$ $\Longrightarrow$ curve F in $\mathbb{R}^3$

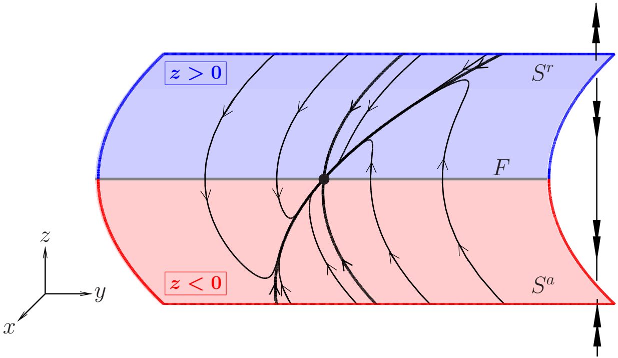
Slow flow

- In order to understand the flow on $S := \{y - x^2 = 0\}$:
  - diff. w.r.t time of $y - x^2 = 0$ gives
  $$y_t - 2xx_t = 0$$
  - projection onto the (x, z)-plane
  $$\begin{aligned} 2xx_t &= a - x \\ z_t &= \mu \end{aligned}$$

Problem along the fold curve F: $x = 0$ $\Rightarrow$ system is singular

- desingularisation (time rescaling) gives
$$\begin{aligned} x_t &= a - x \\ z_t &= 2x\mu \end{aligned}$$
- defines the slow flow

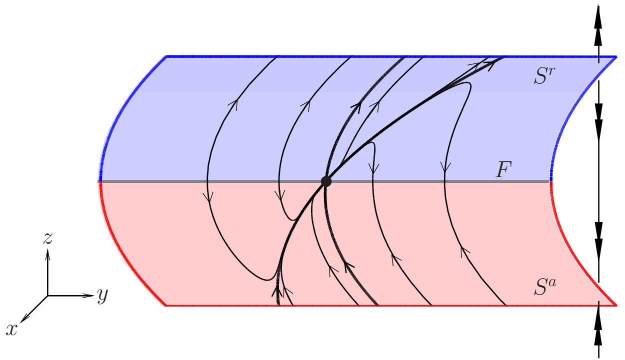
Flow of the desingularised reduced system



Remarque: Equilibrium point on the fold curve!!

The eigenvalue ratio at this point is a function of μ

Flow of the reduced system



This point is called a **folded node**

Singular canards

- Equilibria of the desingularised reduced system are on the fold curve
- This defines folded-singularities (folded-*node*, folded-*saddle*, folded-*focus*, ...)
- time rescaling to desingularise the reduced system
  - solutions crosses the origin with $\neq 0$ speed
  - such orbits are called (singular) canards

$$\boxed{\textbf{Situation for } \varepsilon > \mathbf{0} \textbf{ ?}}$$

The perturbed case: $\varepsilon > 0$

$\boxed{\textbf{Recall}}$ Outside F, Fenichel theory ensures the existence of attracting and repelling slow manifolds S_ε^a of dim. 2. and S_ε^r

- Along the fold curve, one cannot apply these results anymore; however, one can follow $S_\varepsilon^{a,r}$ by the flow.

- Their transversal intersections define maximal canards!

- Many of them can appear in this 3-dim. context!

- Differentes techniques of analysis: nonstandard analysis, matched asymptotics, parameter blow-up, . . .

The perturbed case: $\varepsilon > 0$ THEORY

Benoît (2001)

when $\mu \notin \mathbb{N}$, two (primary) maximal canards (primaires) exist for every ε (only one if $\mu \in \mathbb{N}$)

Szmolyan, Wechselberger (2001)

same results proven with a different method

Wechselberger (2005)

when $\mu \in 2\mathbb{N}+1$, bifurcations of primary canards occur and they give rise to secondary canards. If $\mathrm{int}(\mu) = 2k + 1$ then there exist k secondary canards, corresponding to $2k + 1$ twists of the slow manifolds around each other.

Good news: Canards are robust in $\mathbb{R}^3$, they exist for $O(1)$ ranges of parameters

The perturbed case: $\varepsilon > 0$ NUMERICS

- a 2D invariant manifold can be represented by a family of orbit segments

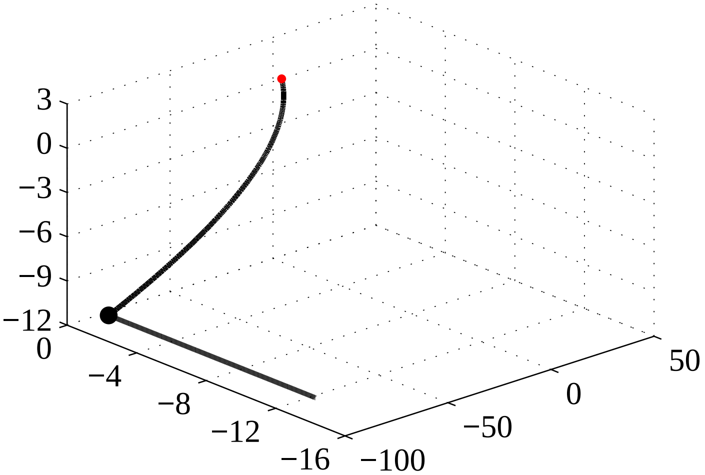
- this family is obtained by **numerical continuation** of a one-parameter family of **well-posed Boundary-Value Problems (B.V.P.)**

- Our numerical protocol imposes that each orbit segment has

  - its initial point on a curve traced on S (we make use of the **normal hyperbolicity** of S outside the fold curve!)

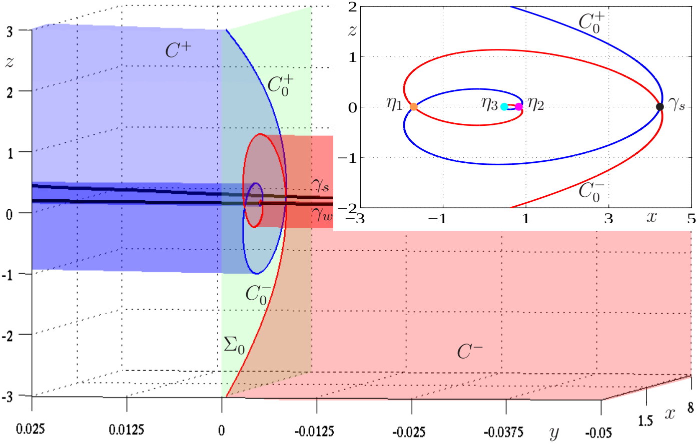
  - its end point in a cross-section transerval to the flow near the fold, chosen so that the resulting surface renders a piece of interest of the slow manifold

Illustration: numerical computation of S^a_ε

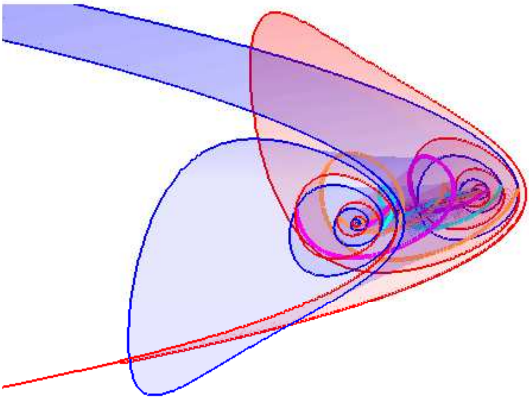


[*Desroches, Krauskopf et Osinga, SIAM J. Appl. Dyn. Sys.* **7***(4), 2008*]

Slow manifolds and canards



Interactions on each side of the folded node



Example: the self-coupled FitzHugh-Nagumo system

(see demo `fnc` of AUTO)

$$
\boxed{\begin{aligned}
v_t &= h - (v^3 - v + 1)/2 - \gamma s v \\
h_t &= -\varepsilon(2h + 2.6v) \\
s_t &= \beta H(v)(1-s) - \varepsilon\delta s
\end{aligned}}
$$

variables	
v	membrane potential
h	inactivation of ionic channels
s	synaptic coupling
$H(v)$	Heaviside
parameters	
γ	coupling strength
β	activation of the synapse
$\varepsilon\delta$	inactivation of the synapse
$0<\varepsilon\ll 1$	small parameter

Example: the self-coupled FitzHugh-Nagumo system

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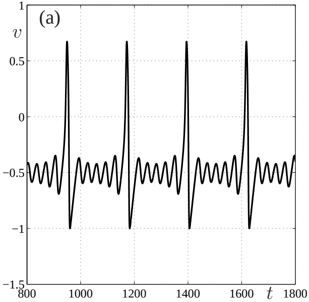

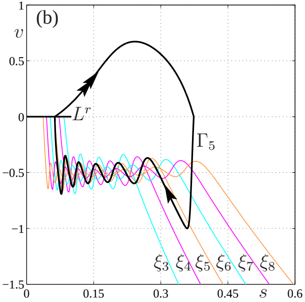

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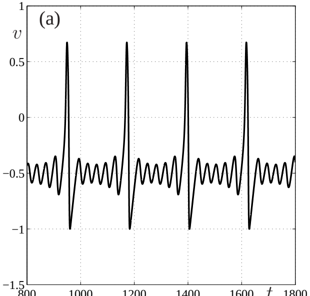
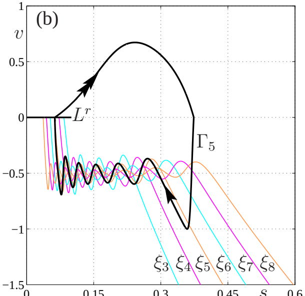




The silent phase system

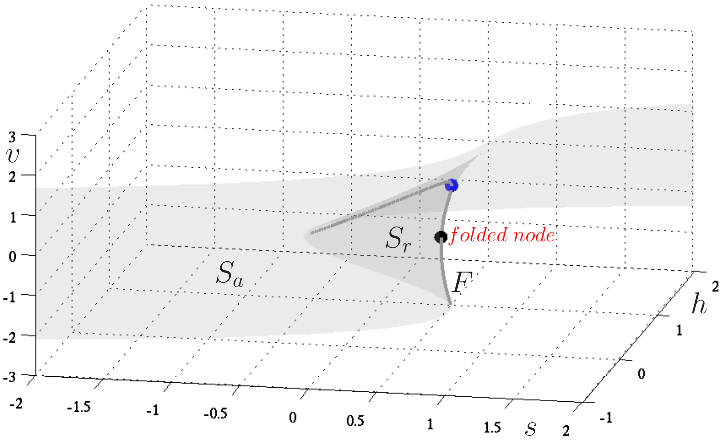
- 3D slow-fast system with 2 slow variables

$$
\begin{aligned}
v_t &= h - (v^3 - v + 1)/2 - \gamma s v \\
h_t &= -\varepsilon(2h + 2.6v) \\
s_t &= -\varepsilon \delta s
\end{aligned}
$$

$$
\begin{aligned}
v < 0 &: \texttt{fast variable} \\
h \in [0,1] &: \texttt{slow variable} \\
s \in [0,1] &: \texttt{slow variable}
\end{aligned}
$$

- `critical manifold`: $S := \{h = (v^3 - v + 1)/2 + \gamma s v\}$
- S possesses
  - a `fold curve` $F := \{h = \frac{1}{2} - v^3, s = \frac{1-3v^2}{2\gamma}\}$
  - a cusp point $(v, h, s) = (0, \frac{1}{2}, \frac{1}{2\gamma})$
- the fold curve F separates S into an attracting sheet S_a from a repelling one S_r

The critical manifold S and the fold curve F



Folded node & return mechanism

- "FHN+self-coupling" possesses a folded node for certain values of param. γ and δ
  - `(singular) canards` in the reduced system
  - perturbation ($\varepsilon > 0$) $\Rightarrow$ canard solutions of the silent phase system
- the active phase offers a *return mechanism* that can generate complex periodic solutions, usually referred to as `mixed-mode oscillations (MMOs)`

$\Rightarrow$ **definition:** periodic solutions formed by an alternation of small-amplitude oscillations and large-amplitude oscillation; notation: ℓ^s for s small oscillations and ℓ large.

- **Théorème (Brøns *et al.* 2006):** A slow-fast system in $\mathbb{R}^3$ possessing 2 slow variables, a folded node and a return mechanism produces MMOs of type 1^s; the number s of small osc. is determined by the canard solutions associated with the folded node.

Computation of perturbed manifolds: starting solution?

Importante question : how did I get the initial segment ?

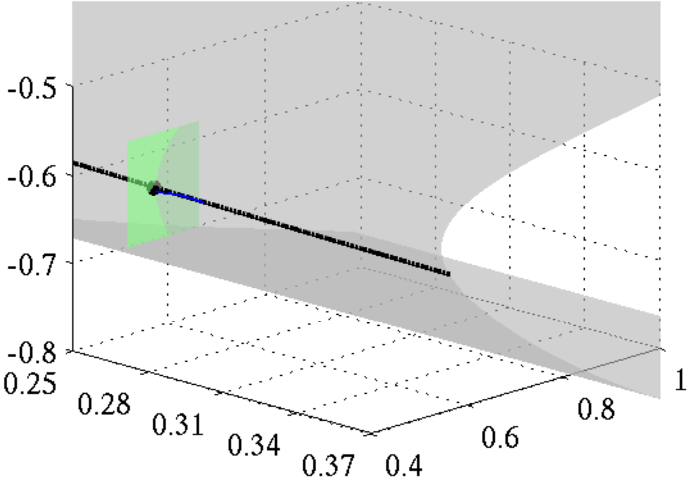
Computation of perturbed manifolds: starting solution?

- **Present case:** easy! This system is of "normal form" type, it possesses enough symmetry and explicit solutions
- **In general:** Not that easy!!

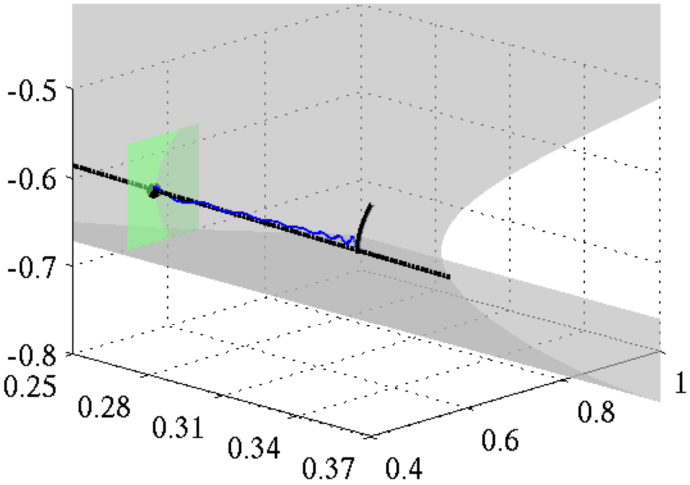
$$\boxed{\textbf{A good solution...}}$$

- **Direct simulation?** Delicate! Near the fold curve F, the competition between time scales is very strong $\Rightarrow$ substantial chances to be ejected!
- **Continuation of families of BVPs** based on an (additional) **homotopy method**

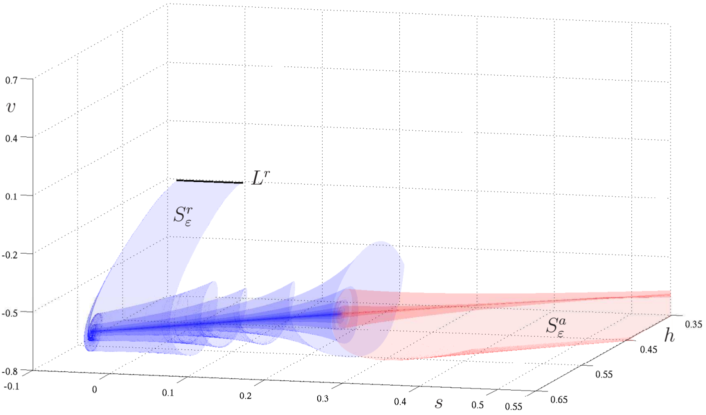
Step 1: away from the folded node . . .



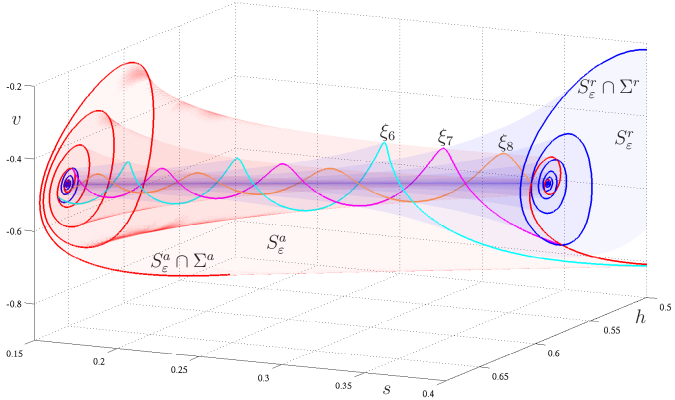
Step 2: away from the fold curve ...



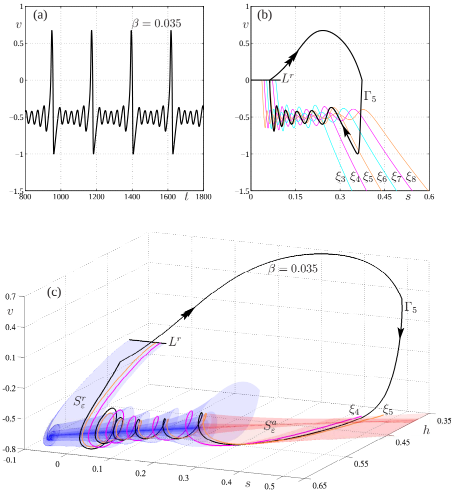
The slow manifolds



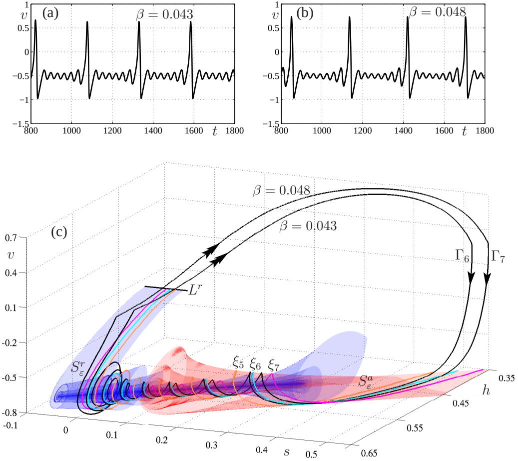
Interactions between S_ε^a and S_ε^r



MMOs rotation sectors



MMOs rotation sectors



[Desroches, Krauskopf et Osinga, *CHAOS* **18***(1)*, 2008]

Numerical Continuation

Slow-fast systems and canards in $\mathbb{R}^2$

Slow-fast systems and canards in $\mathbb{R}^3$ with 2 slow variables

Computation of 2D slow manifolds and canards in $\mathbb{R}^3$

Detecting and “continuing” canards

Conclusion

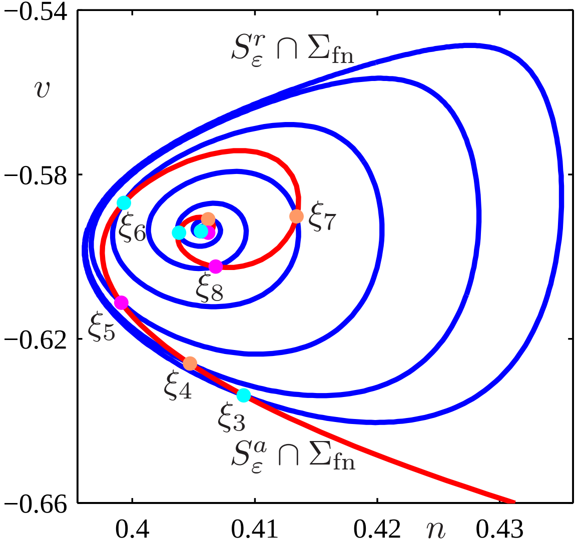
Detection of canard orbits: method

- we consider the intersection curves $S_\varepsilon^a \cap \mathbf{\Sigma}$ and $S_\varepsilon^r \cap \mathbf{\Sigma}$ of the slow manifolds with a common end section $\mathbf{\Sigma}$
- approximation of the coordinates of these $\cap$ points in $\mathbf{\Sigma}$

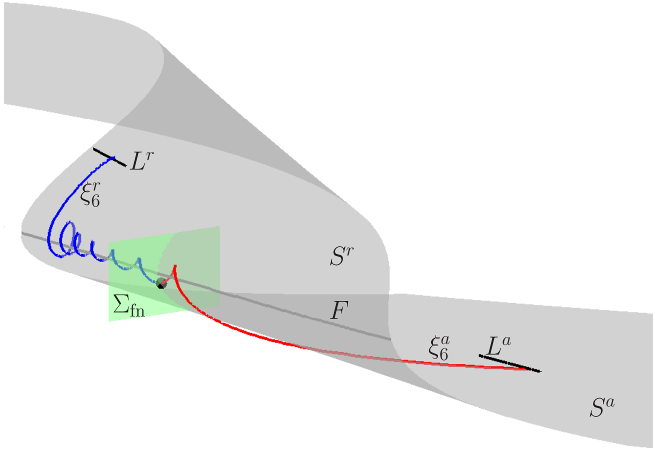
  $\rightsquigarrow$ correspond to **canard orbits**!

- stop the computations at the corresponding values using test-fonctions
- this yields two "half canard segments": ξ^a on S_ε^a and ξ^r on S_ε^r
- ensuring that $|\xi^a(1) - \xi^r(0)|$ is suff. small, a simple Newton iteration converges towards a "full" canard segment.

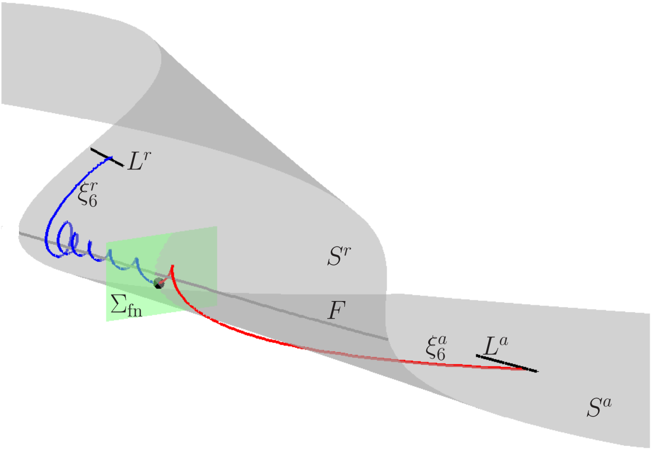
Detection of canard orbits: in section Σ_{fn}



Detection of canard orbits: in $\mathbb{R}^3$



"Continuing" canards: initial situation

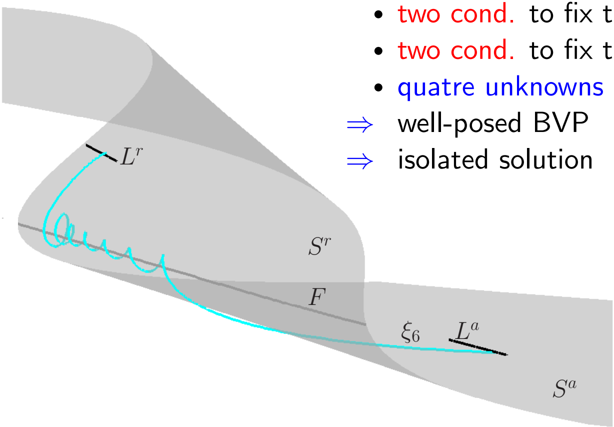


“Continuing” canards: no need for Σ_{fn}!

- two cond. to fix the ini. cond. on L^a
- two cond. to fix the end point on L^r
- quatre unknowns (3 var. + int. time)

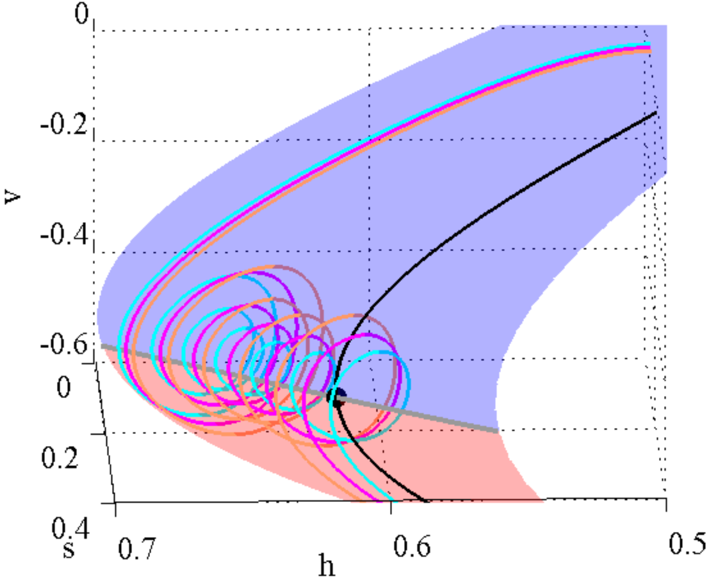
$\Rightarrow$ well-posed BVP

$\Rightarrow$ isolated solution

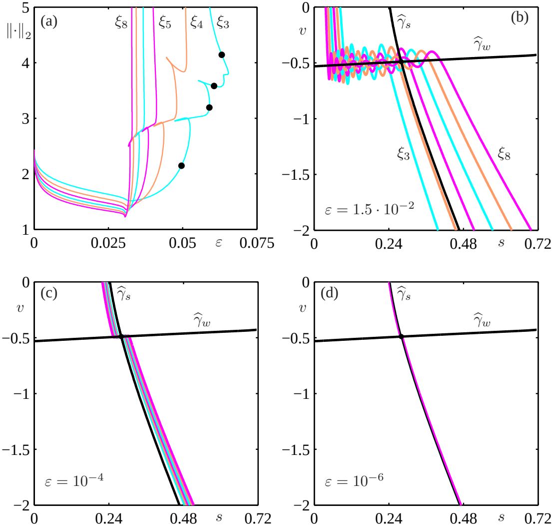


varying one system parameter $\Rightarrow$ **one-parameter family of canard orbits**

FHN example: continuation in ε for $\varepsilon \to 0$

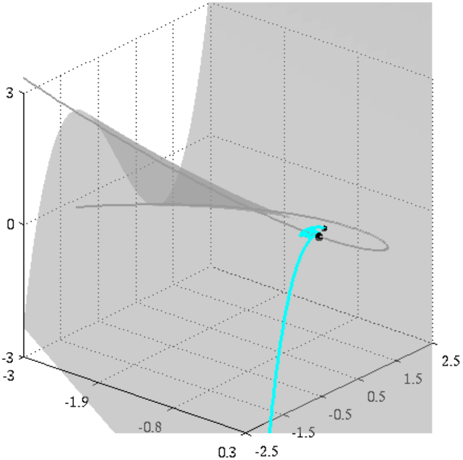


FHN example: bifurcation diagram as a function of ε

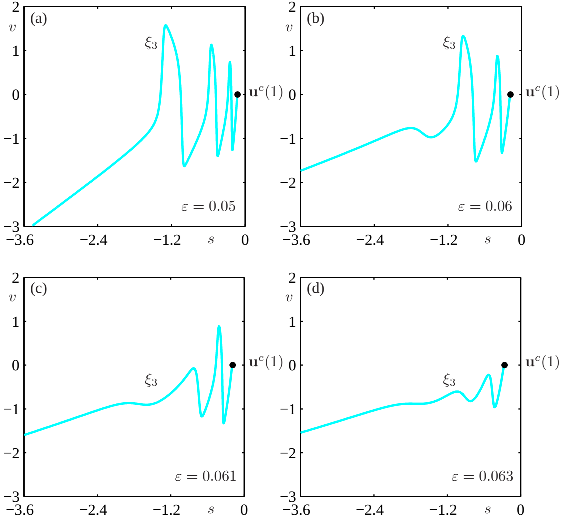


[*Desroches, Krauskopf et Osinga, Nonlinearity* **23***(4): 739–765, 2010*]

FHN example: canard continuation in $\varepsilon \nearrow$



Solution profile when ε is increased



[*Desroches, Krauskopf et Osinga, Nonlinearity* **23***(4): 739–765, 2010*]

Summary for the 3D case: canards/folded node

- canards like to live in $\mathbb{R}^3$!
- still challenging numerically **BUT** the continuation of BVPs proves to be very efficient and reliable to
  - compute slow manifolds
  - detect their transversal intersections: maximal canards
  - continue canards in any system parameter
- folded node type canards have specific small oscillations
- **add a return mechanism:** they organise the dynamics of families of complex oscillatory solutions Mixed-Mode Oscillations (MMOs)
- the underlying theory is recent and still under development (Wechselberger, Krupa, Popovic, Guckenheimer, ...)

$\boxed{\text{one could say, of course, much more ...}}$

What I haven't talked about . . .

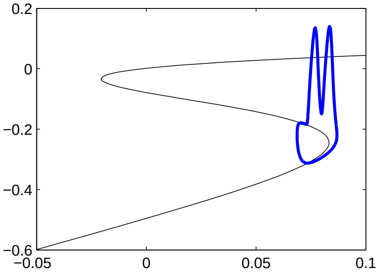
- **bifurcations of folded singularities** give rise to much more complicated dynamics and much more complex MMO patterns

$\Rightarrow$ theory is still incomplete (folded saddle-node singularity, singular Hopf bifurcation, . . .)

- I talked about the **"1 fast - 2 case"** but there is (at least) one other very interesting case in 3D slow-fast systems: **"1 slow - 2 fast"**

$\Rightarrow$ can give rise to bursting oscillations, possibly linked to the canard phenomenon!

Spike-adding “via canards”!



This is a Morris-Lecar type system in $\mathbb{R}^3$ est obtained by putting a slow dynamics onto the applied current I

Bifurcation parameter: ε

Additional references