An example from chemical dynamics

The model

- Peroxidase-Oxidase (PO) reaction

- we consider the Olsen model (1983):

$$
\begin{aligned}
A' &= -k_3ABY + k_7 - k_{-7}A \\
B' &= \alpha(-k_3ABY - k_1BX + k_8) \\
X' &= k_1BX - 2k_2X^2 + 3k_3ABY - k_4X + k_6 \\
Y' &= -k_3ABY + 2k_2X^2 - k_5Y
\end{aligned}
$$

- “classical” parameter values:

k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_{-7}	k_8	α
0.28	250	0.035	20	5.35	0	0.8	0.1	0.825	1

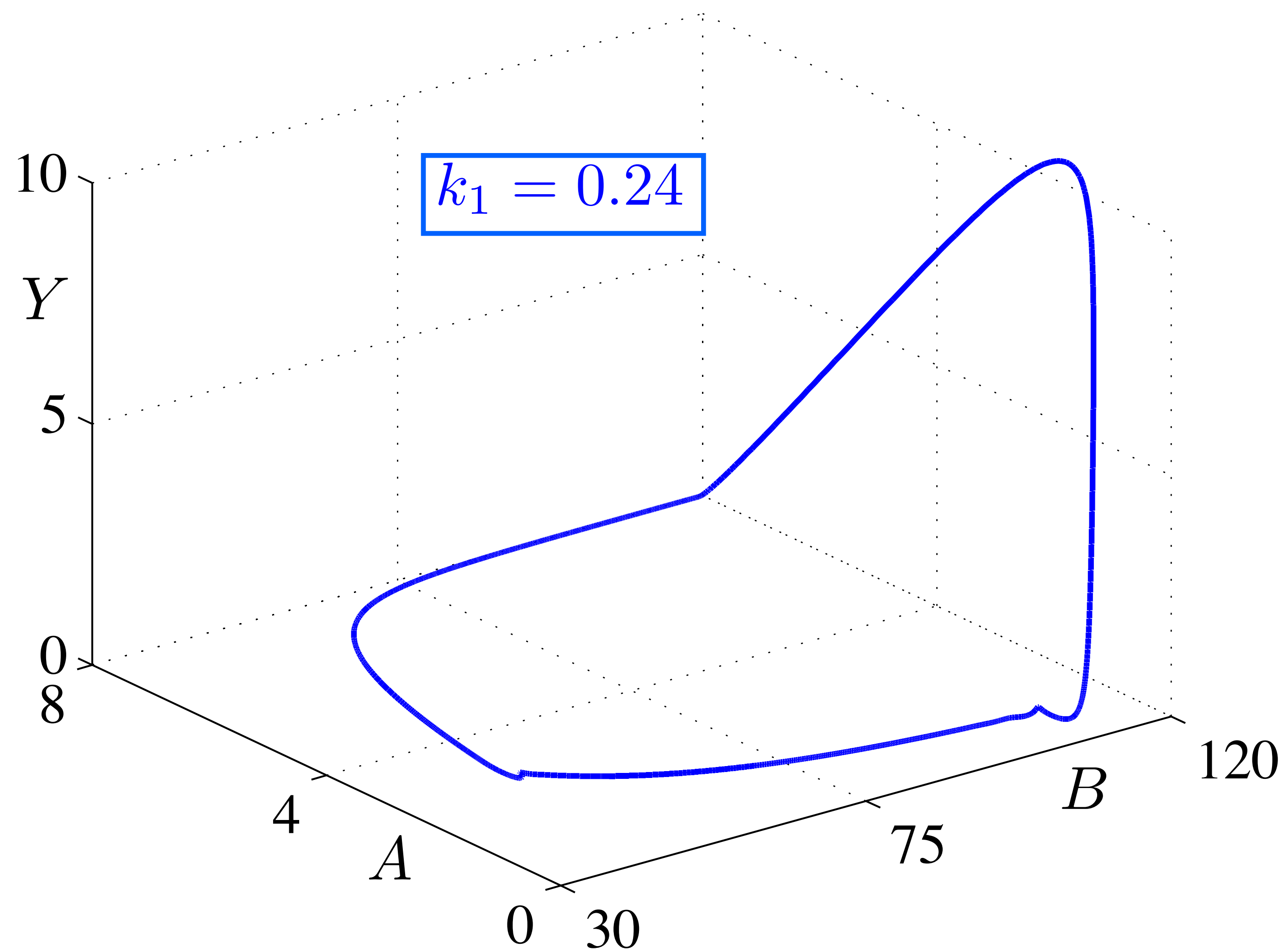
- one should have: $0 < k_6 \ll 1$ but from math. viewpoint, interesting to consider the limit: $k_6 = 0$ since the (A,B)-plane becomes invariant.

- α : artificial time scale parameter ... we will see the reason soon ...

The Olsen model's typical dynamics

- multiple time scales, in particular B seems to evolve on slower time scale than the other 3 variables A, X, and Y.

- direct simulation shows mixed-mode oscillations (MMOs) with different patterns upon variation of several parameters, in particular k_1 .



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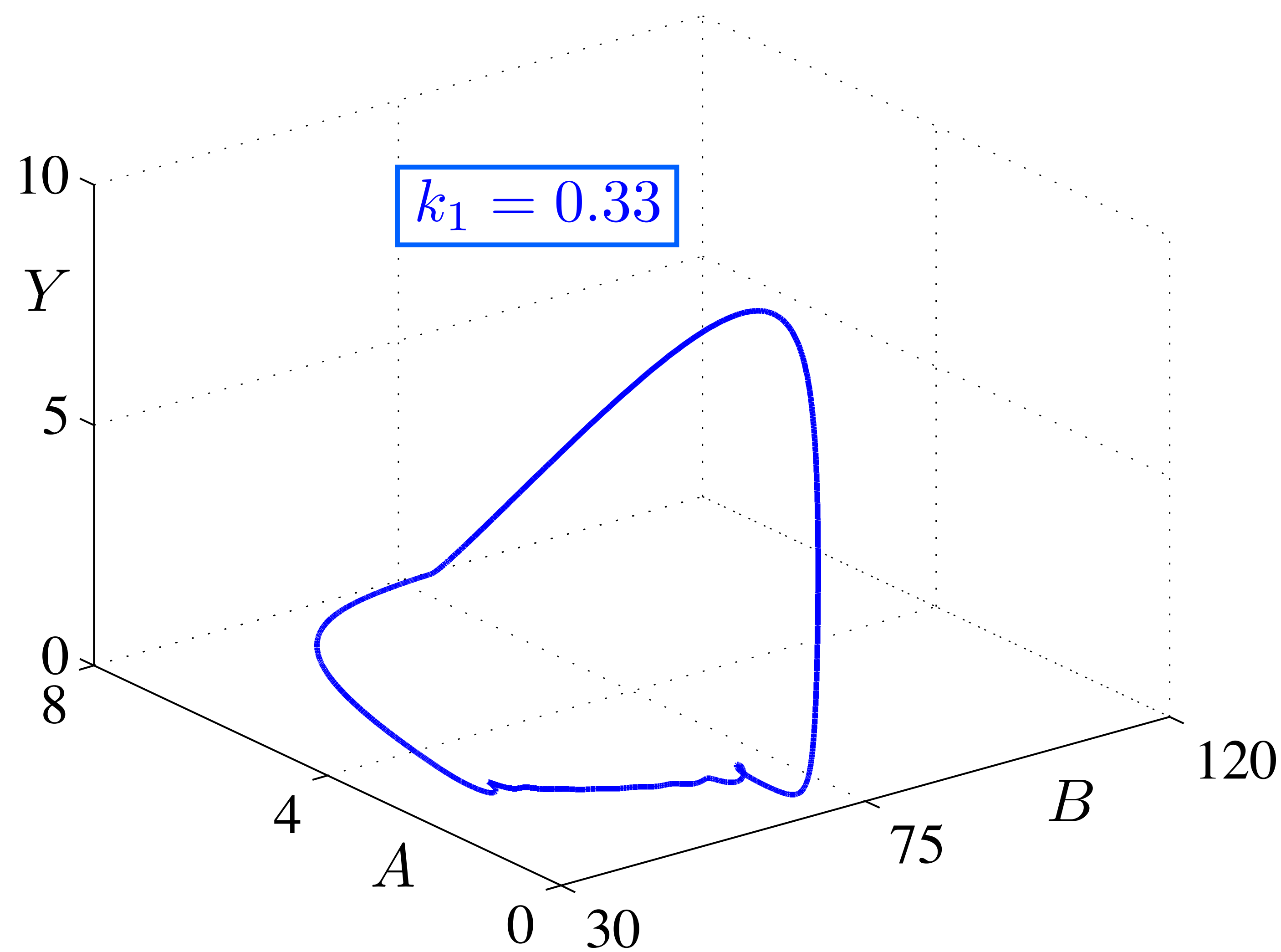

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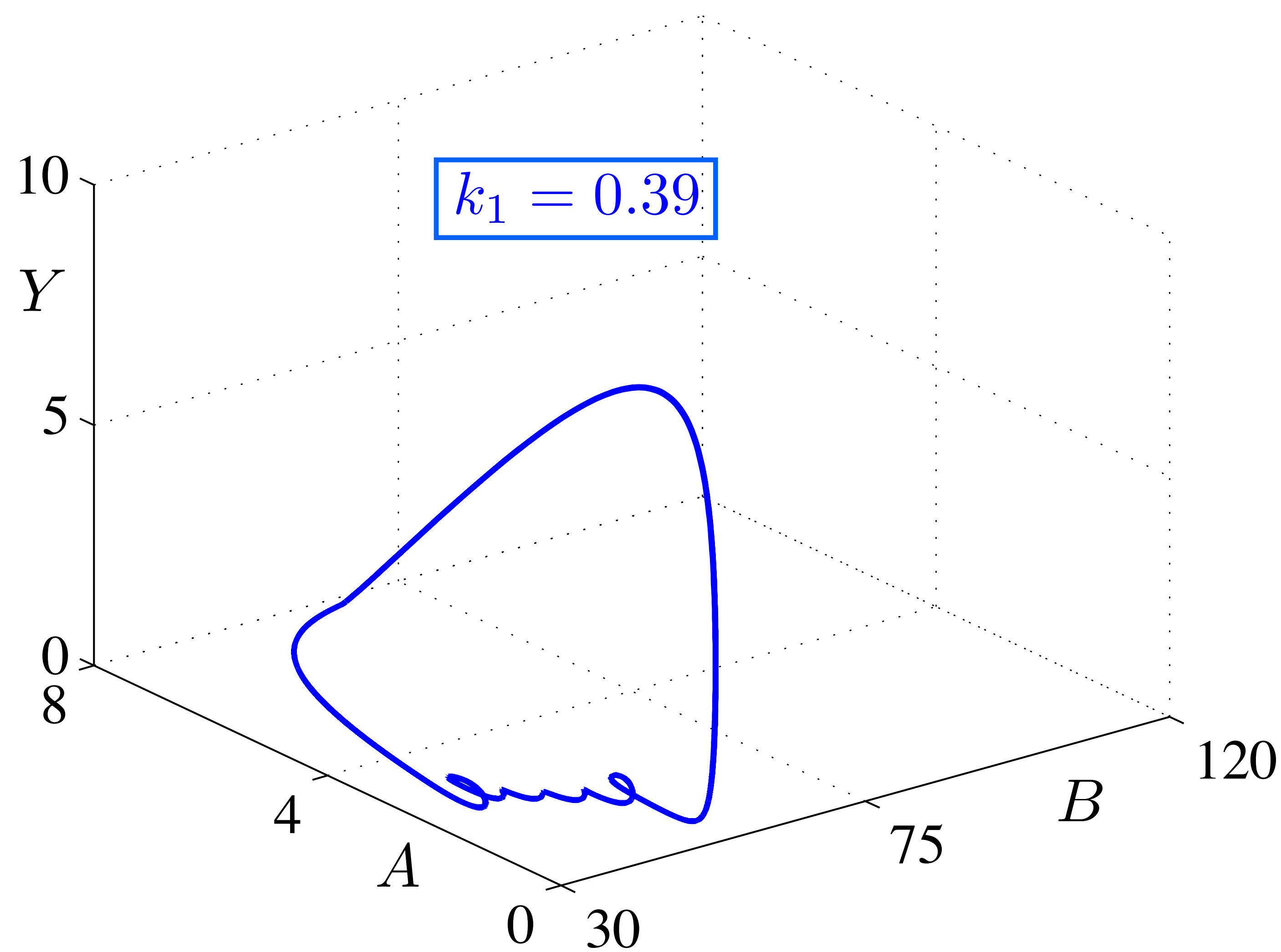


Fast subsystem of the Olsen model

- let's assume that B is a slow variable (seems reasonable from the simulations)
- the fast subsystem is obtained by setting α=0 in the full system:

$$
\begin{aligned}
A' &= -k_3ABY + k_7 - k_{-7}A \\
X' &= k_1BX - 2k_2X^2 + 3k_3ABY - k_4X + k_6 \\
Y' &= -k_3ABY + 2k_2X^2 - k_5Y
\end{aligned}
$$

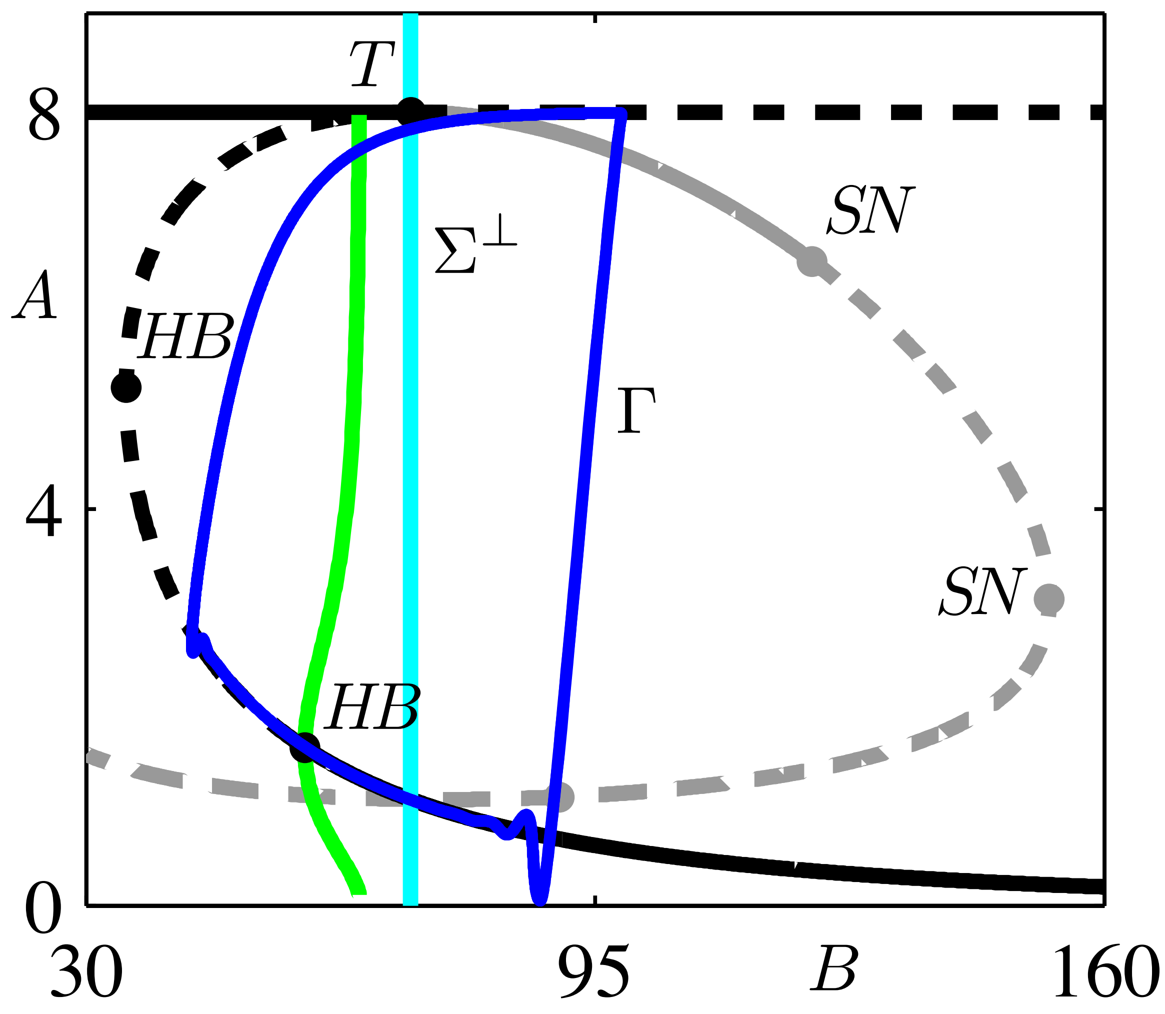
where B is now a parameter!

- **slow-fast dissection:** the dynamics of the full system is dictated by the attractors of the fast subsystem ...

➥ In particular, the passage between 2 different epochs of MMOs correspond to a bifurcation point of the fast subsystem

B slow: good assumption?

- “classical” method: to superimpose the bifurcation diagram of the fast subsystem onto the attractor of the full system ...



- ... it was indeed a good assumption! We start to unravel more the dynamics of the MMOs in the Olsen model.

Mechanisms for MMOs here

- **dynamic bifurcation phenomena :** bifurcation of a fast subsystem that the full system "senses" with a delay, sticking to a repelling branch beyond the bifurcation point

- **Here:** family of focus equilibria of the fast subsystem loses stability at stability at a Hopf bifurcation ... dynamic Hopf bifurcation

➥ **accounts for the small-amplitude oscillations (SAOs) of the MMOs of the full system**

- Delayed exchange of stability of the invariant (A,B)-plane ...

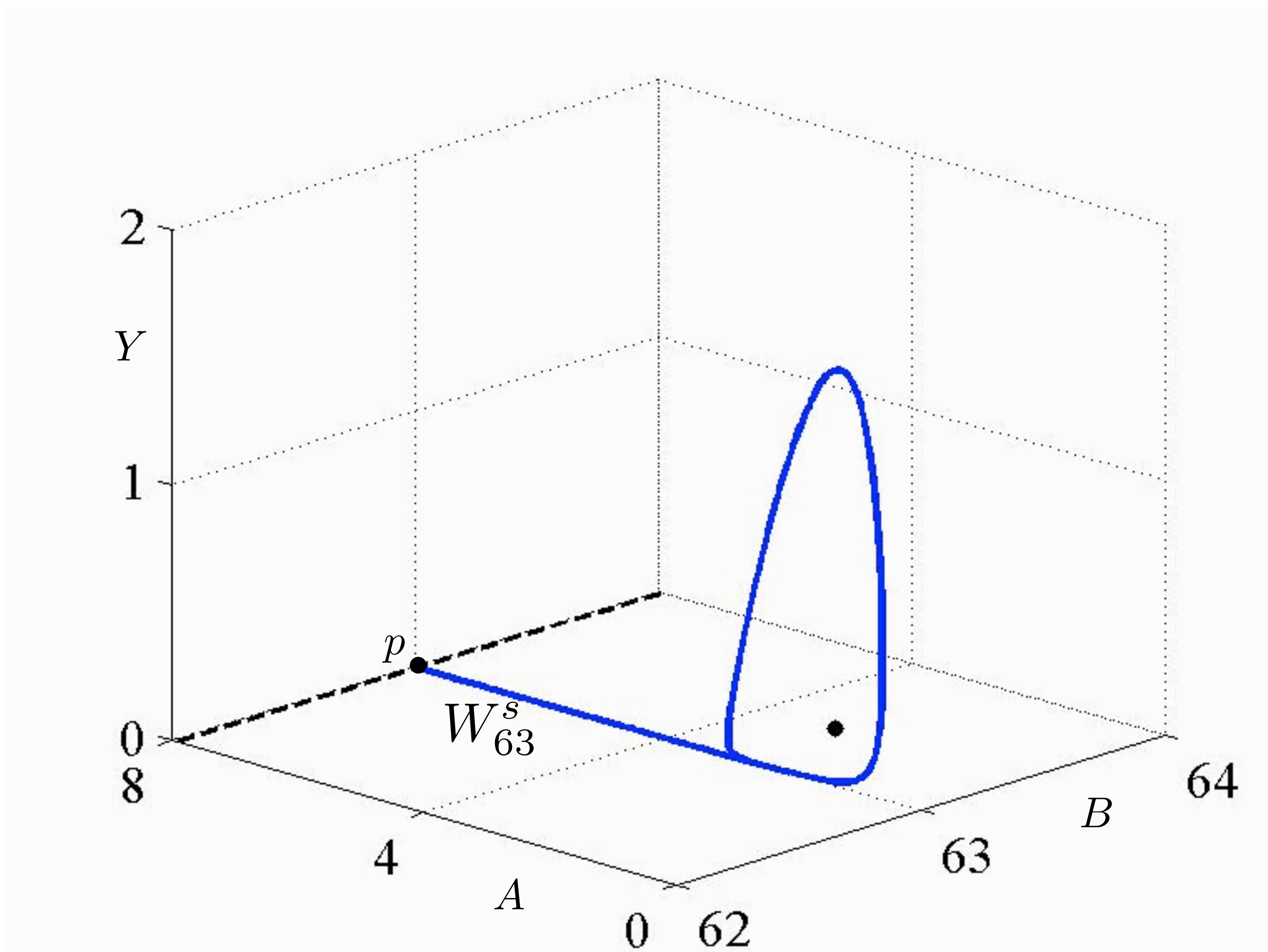
➥ **accounts for the large-amplitude oscillations (LAOs) of the MMOs of the full system**

- Change of # of SAOs ... local interaction between attracting and repelling slow(-like) manifolds

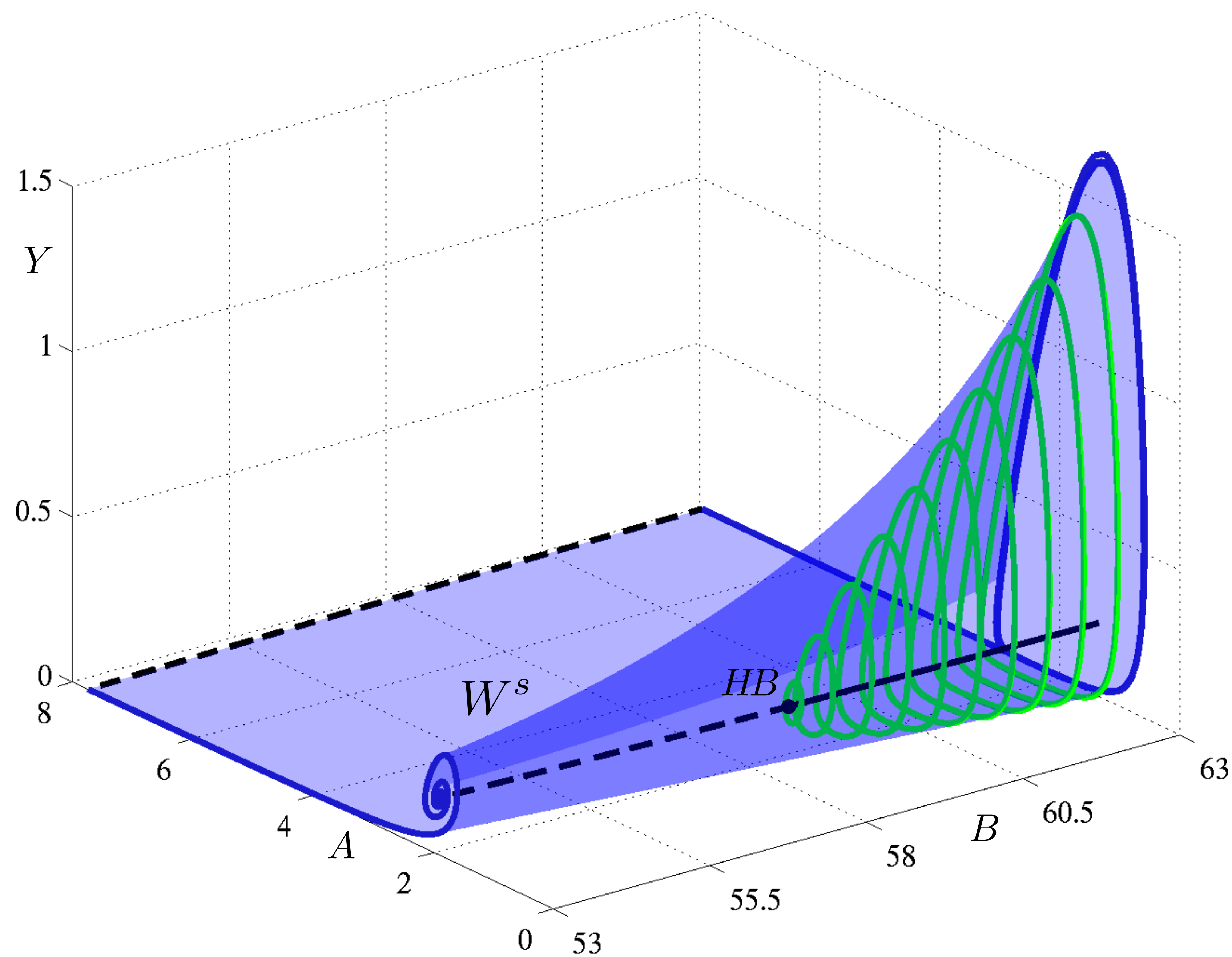
Computation of a 1D stable manifold by numerical continuation

- Boundary conditions:
$$\begin{aligned} \mathbf{u}(0) &\in \{A = k\} \\ \mathbf{u}(1) &\in E^s(p) \end{aligned}$$

- Continuation par: integration time T and k



Repelling slow(-like) manifold

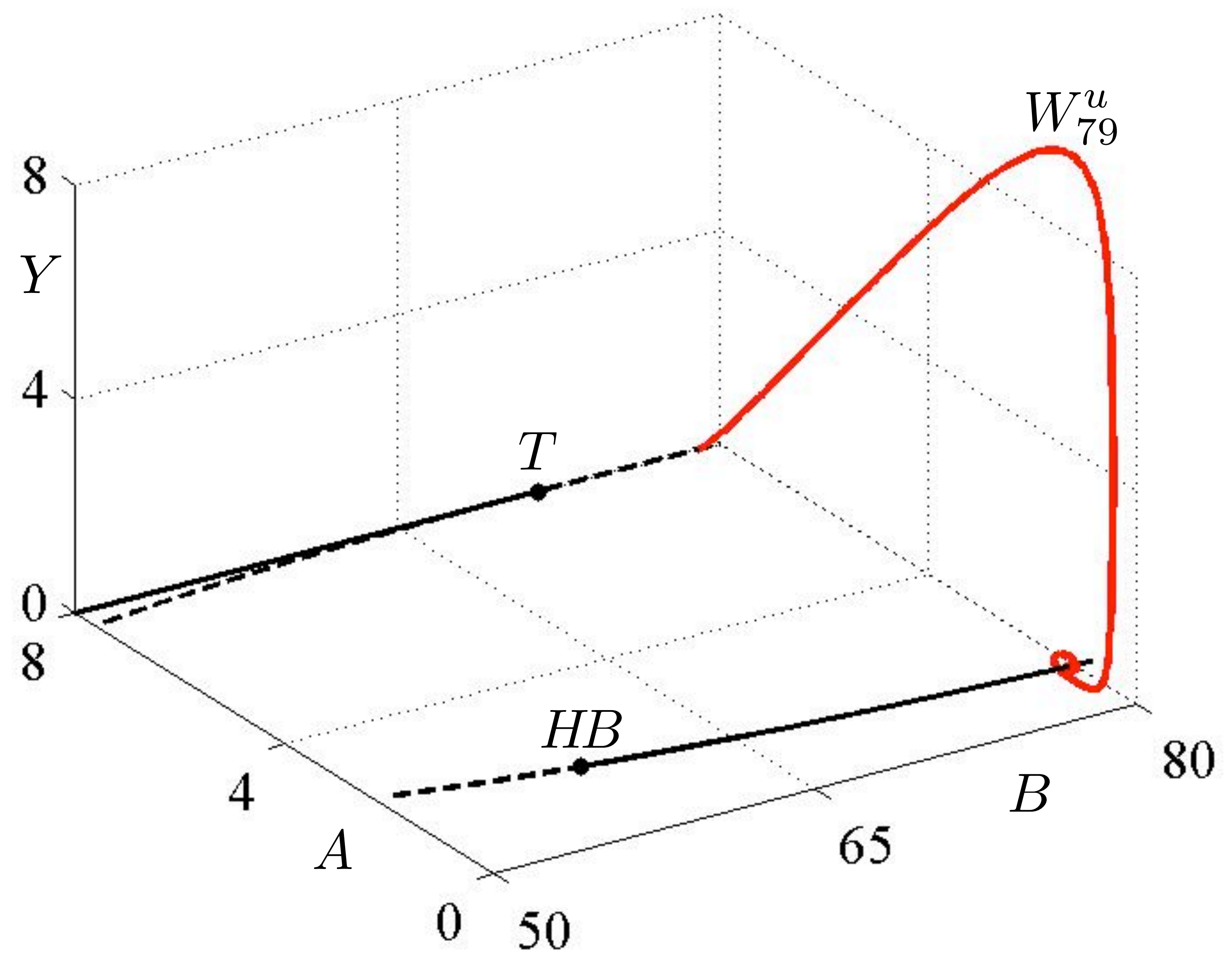


Computation of 1D unstable manifold

- similar boundary conditions as in the repelling case ...

$$\begin{aligned} \mathrm{u}(0) &\in E^u(p) \\ \mathrm{u}(1) &\in \{A = k\} \end{aligned}$$

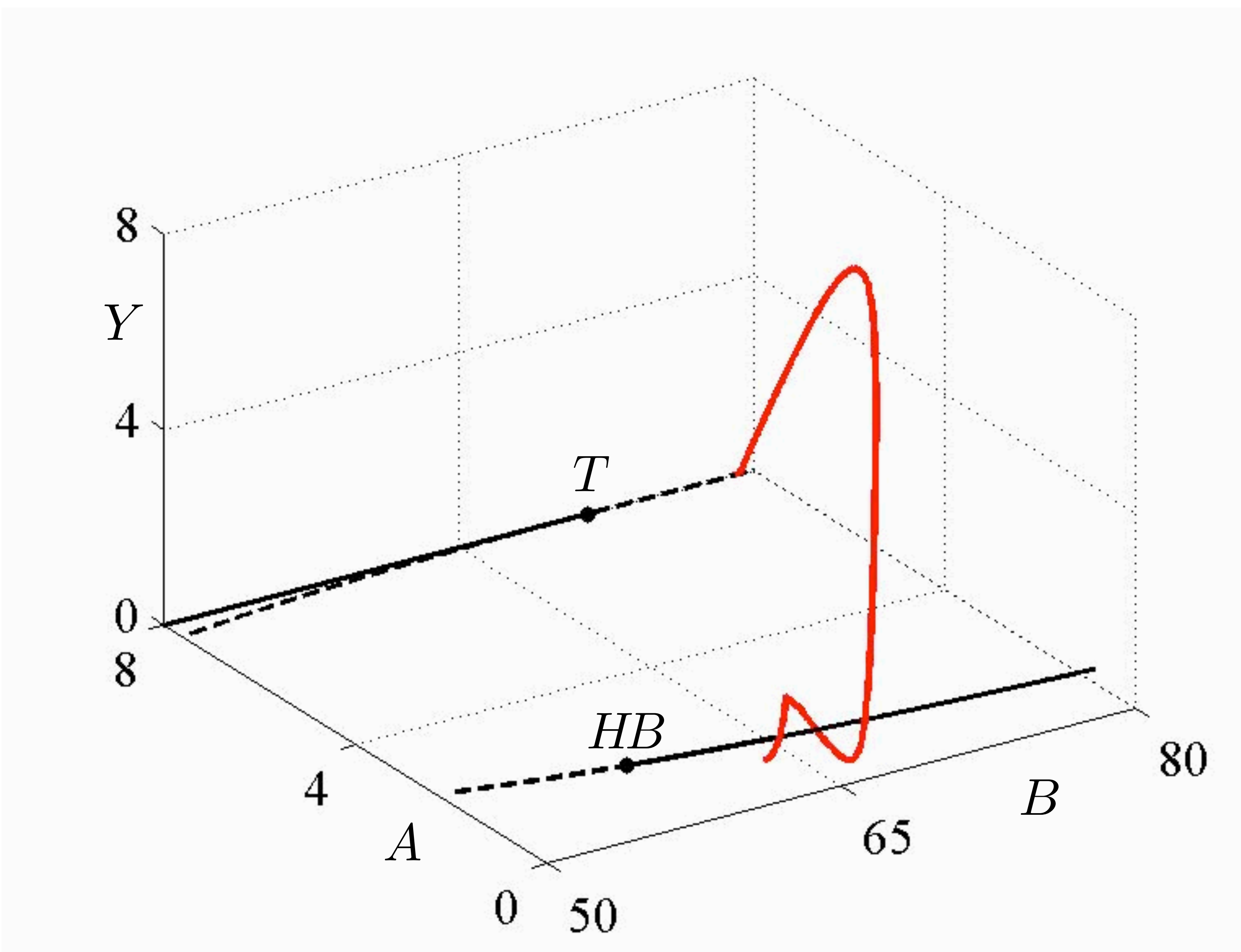
- same continuation parameters: T and k



Attracting slow(-like) manifold: step 1

- "switching on" the dynamics of B by continuing in α the approx. of : W^u_B

fast subsystem α=0 $\Longrightarrow$ α=1 **full system**



5 Boundary conditions:

$$\begin{array}{lll} \mathbf{u}(0) & \in & E^u_{79} \\ \mathbf{u}(1) & \in & \{A = A^*\} \end{array}$$

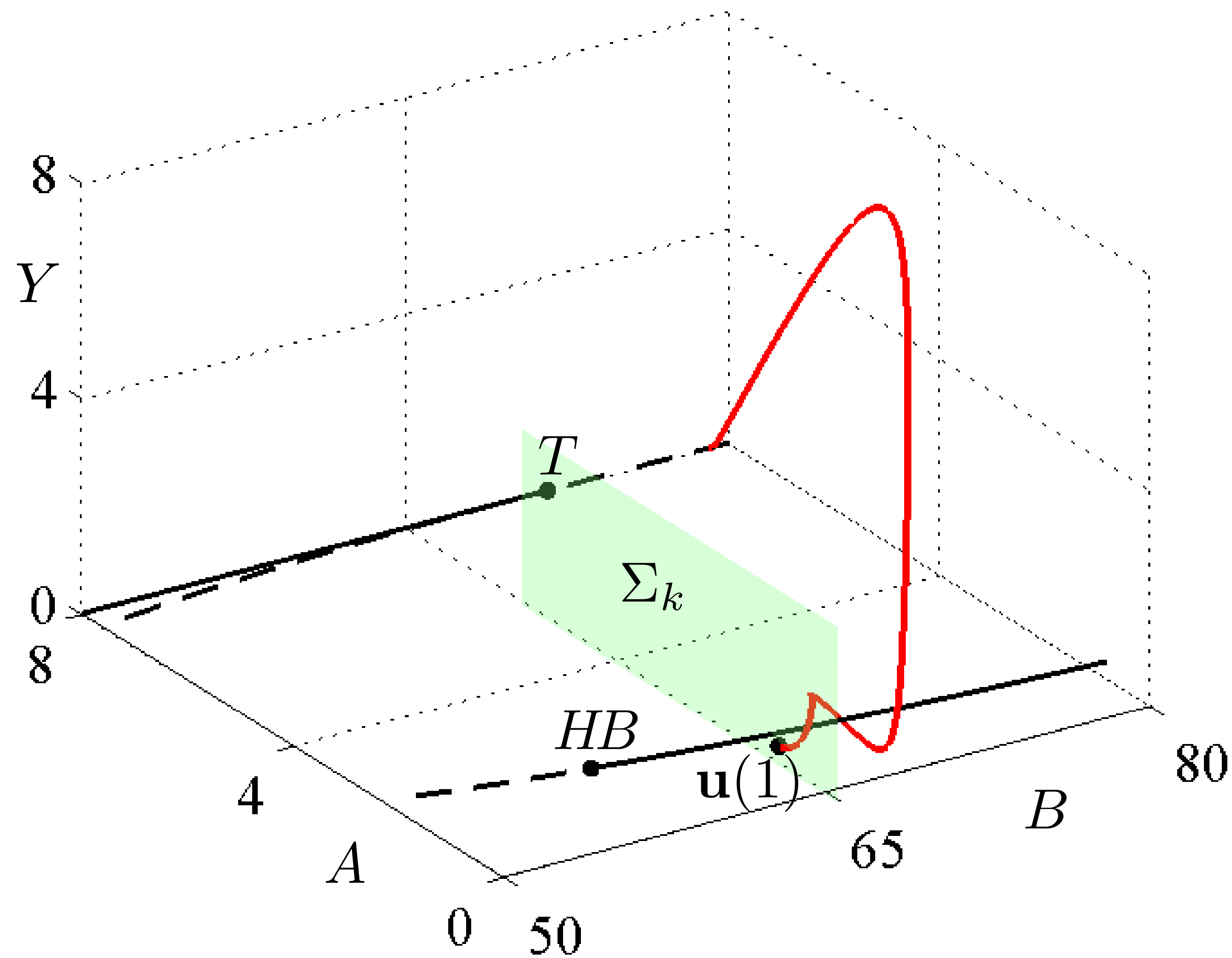
2 continuation par.:

α and T

well-posed!

Attracting slow(-like) manifold: step 2

- "pushing" the B-value of the end point past the Hopf bifurcation point :



5 Boundary conditions:

$$\begin{array}{lcl} \mathbf{u}(0) & \in & E^u_{79} \\ \mathbf{u}(1) & \in & \Sigma_k = \{B = k\} \end{array}$$

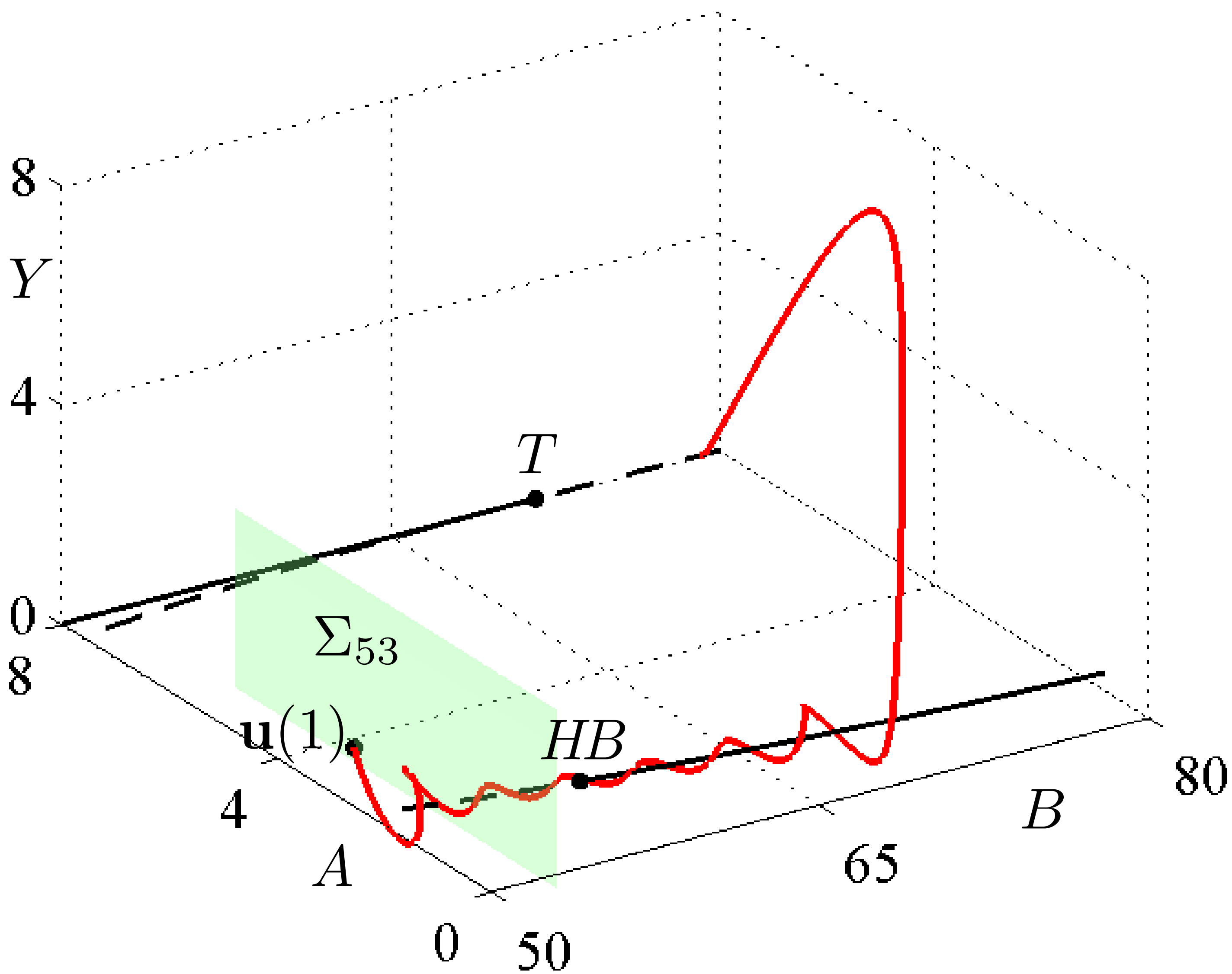
2 continuation par.:

k and T

➥ **well-posed!**

Attracting slow(-like) manifold: step 2

- "pushing" the B-value of the end point past the Hopf bifurcation point :



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\begin{array}{rcl}
\mathbf{u}(0) & \in & E^u_{79} \\
\mathbf{u}(1) & \in & \Sigma_k = \{B = k\}
\end{array}
$$

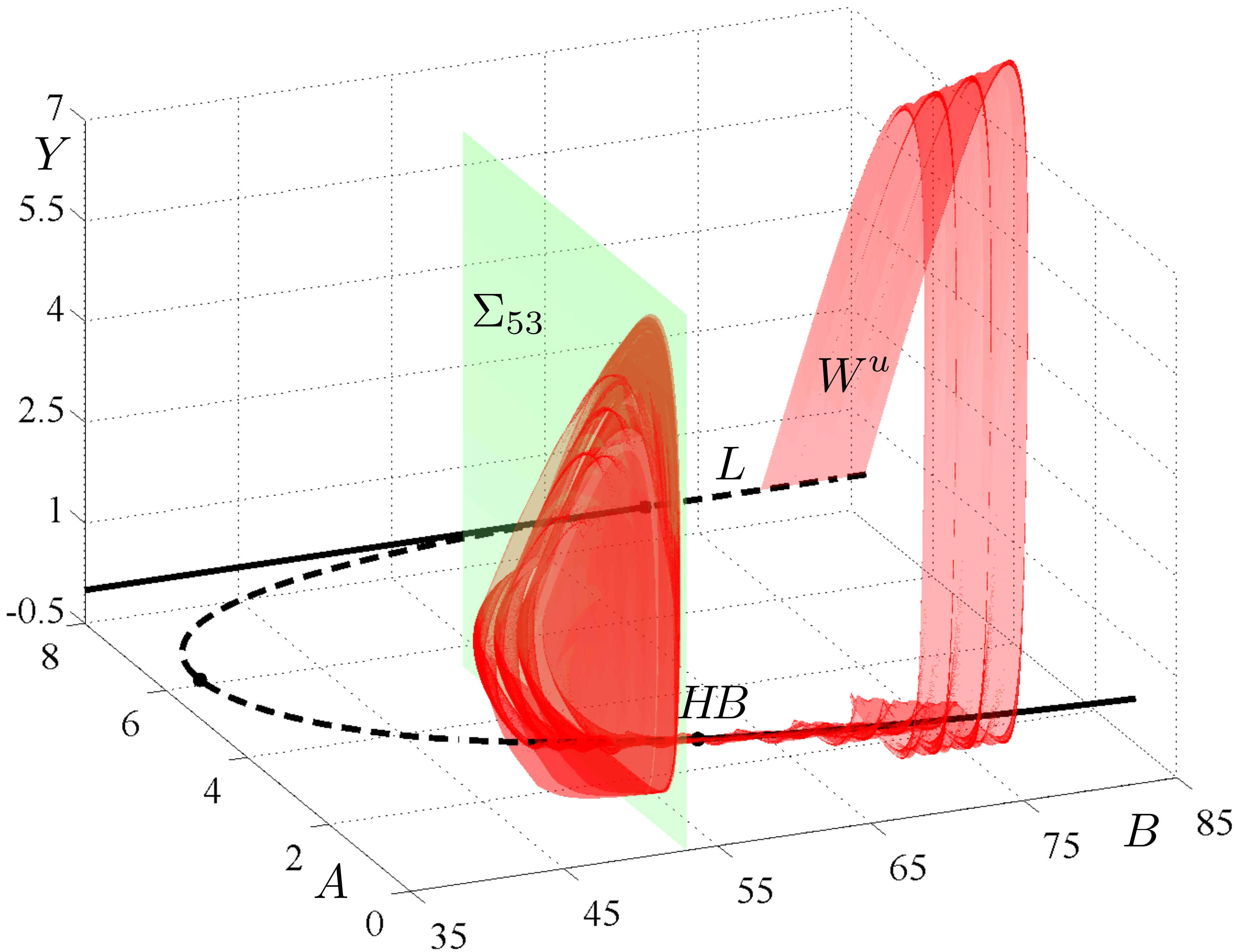
2 continuation par.:

k and T

well-posed!

Attracting slow(-like) manifold: step 3

- move **u**(0) along the trivial branch of saddle equilibria of the fast subsys. :



5 Boundary conditions:

$$\mathbf{u}(0) \in L = \{A = 8, X = Y = 0\}$$
$$\mathbf{u}(1) \in \Sigma_{53}$$

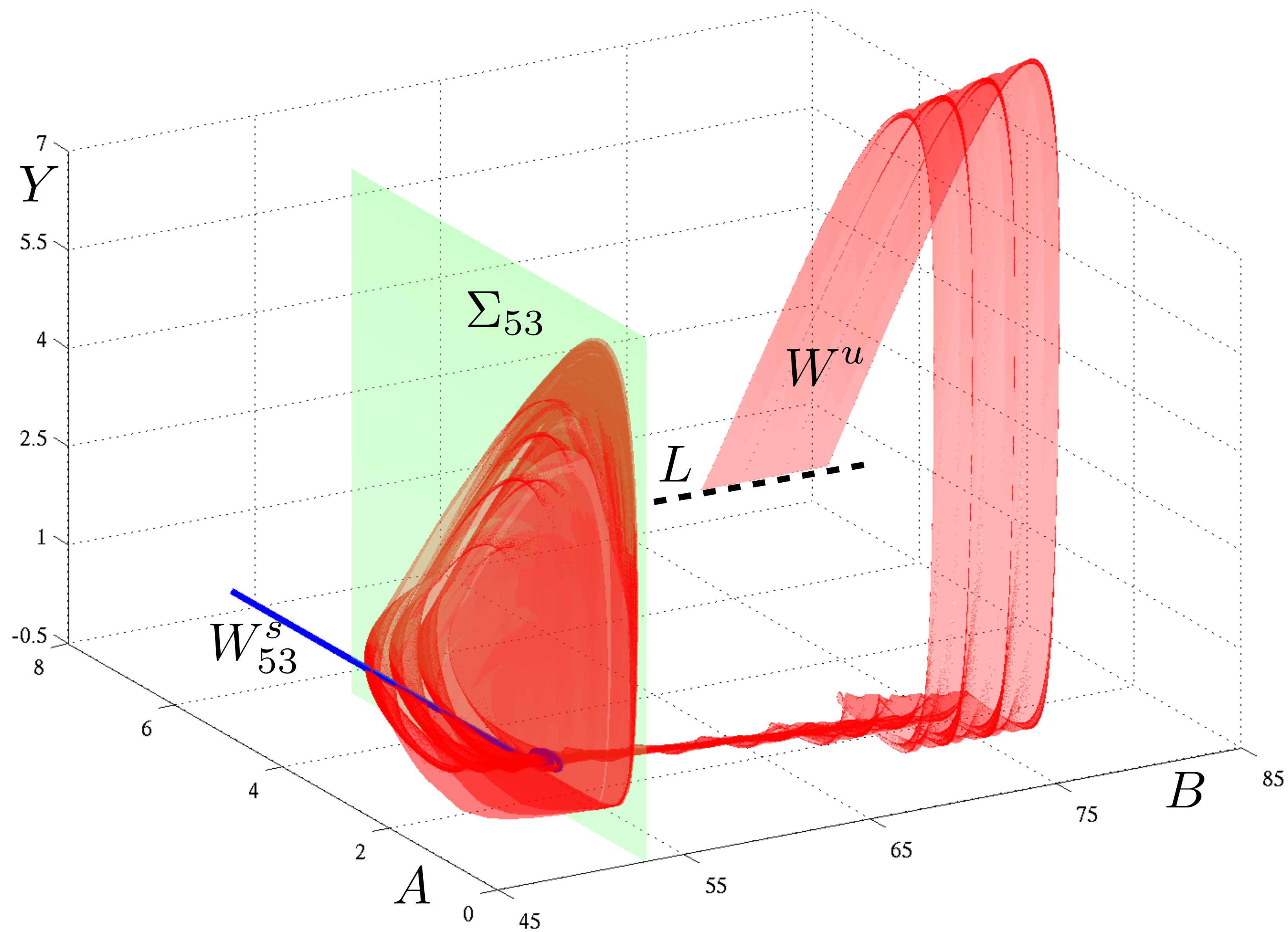
2 continuation par.:

$\boxed{\mathbf{u}_B(0) \text{ and } T}$

➥ **well-posed!**

Local interactions in section {B=53}

- Attracting and repelling extended slow manifold locally twist and intersect transversally ...



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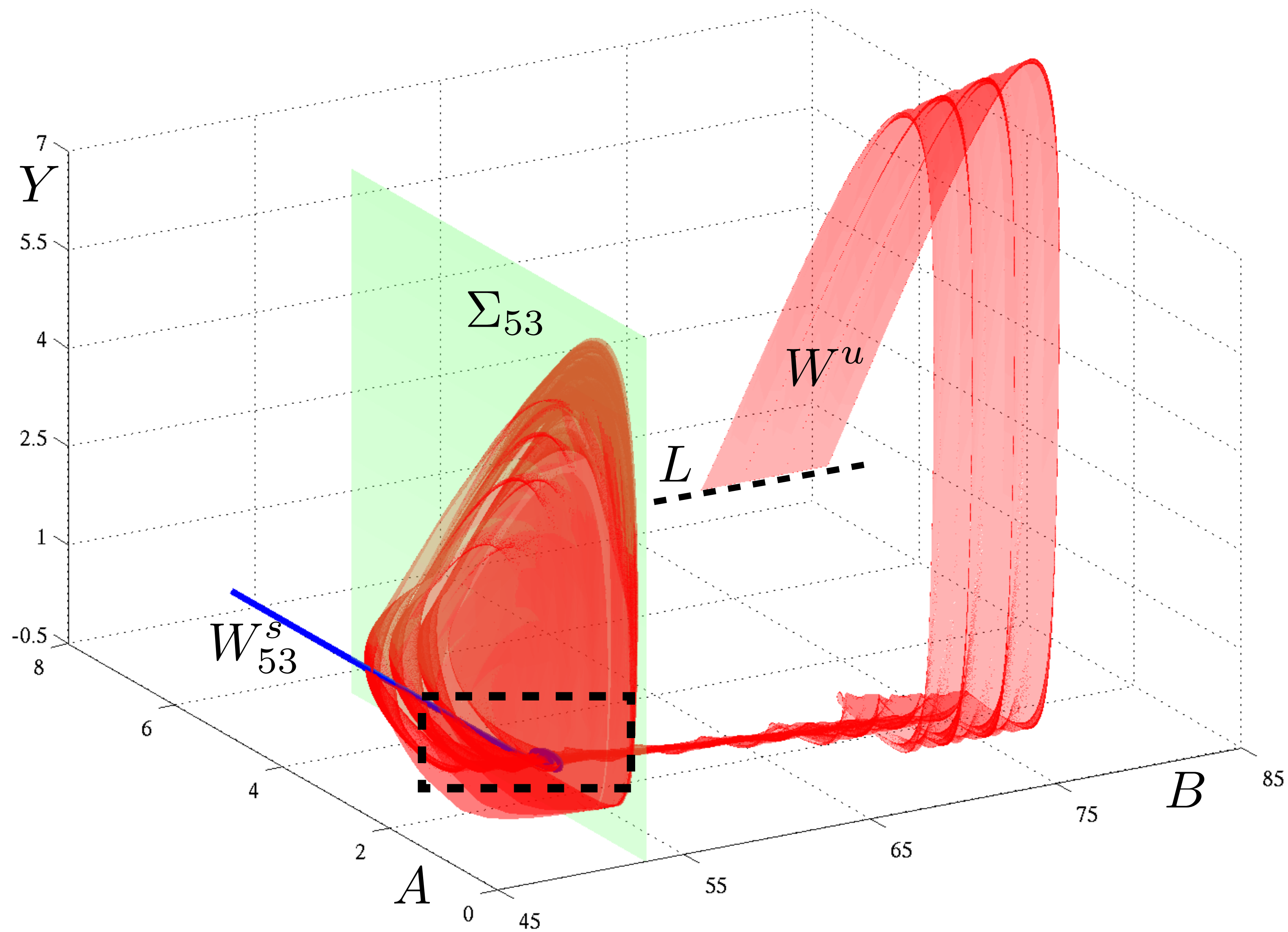

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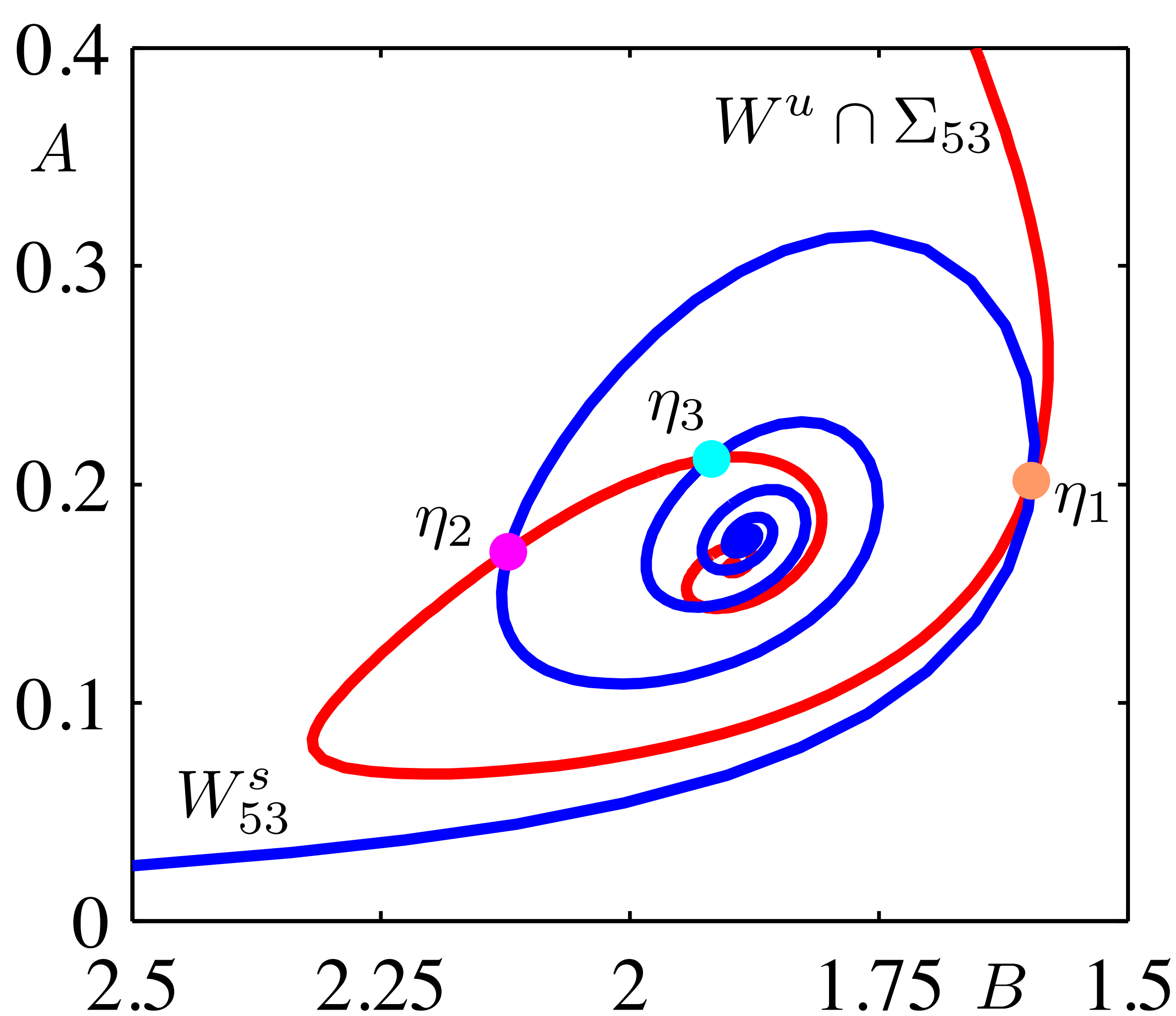
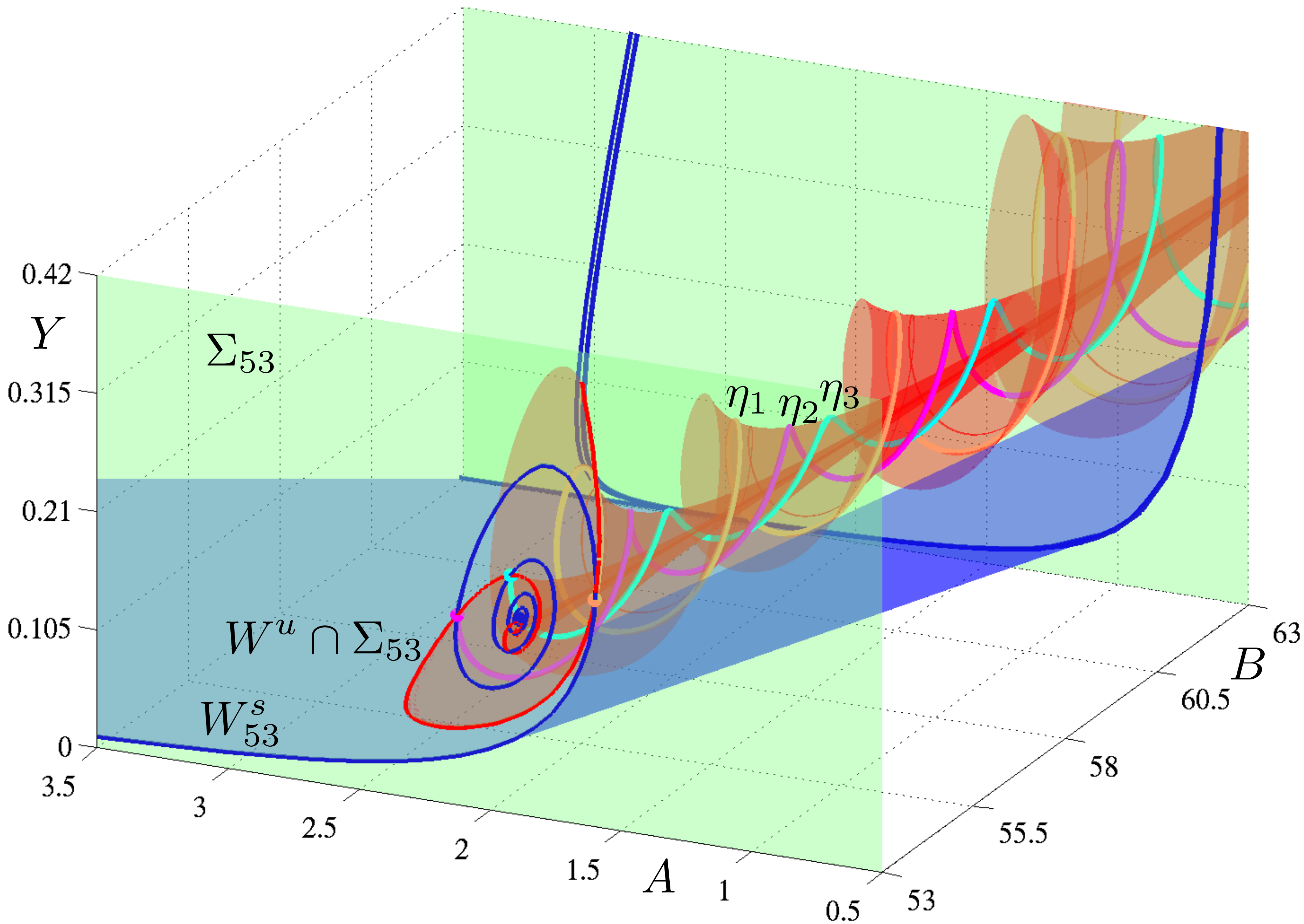


Separating locally MMOs

- Intersection points in section {B=53} correspond to trajectories that organise the transition between diff. patterns of MMOs ...

... n ⇒ n+1 small-ampl. oscillations



[Desroches, Krauskopf & Osinga, DCDS-S **2**(4), 2009]