All volume expanding dynamical systems have positive topological entropy

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Abstract: *Major significance of the topological entropy* – its strong relation to other dynamical invariants such as Lyapunov exponents, topological pressure, fractal dimension, and Hausdorff dimension.
Almost all previous investigations of the topological entropy have been concerned with upper bounds. Exact formulas have been derived under strong smoothness assumptions only. We will give lower bounds of the topological entropy of C^1-smooth dynamical systems on Riemannian manifolds which are sharp in some cases. They are formulated in terms of the phase space dimension and of the exponential growth rates of a singular value function of the tangent map. These rates correspond to the deformation of k-volumes and can for instance be estimated in terms of Lyapunov exponents.

Definition of the topological entropy

Let (X,ρ) be a metric space, $\phi : X \to X$ a continuous mapping.
For $l \in \mathbb{N}$ we introduce the dynamic ball of order l and radius ϵ

$$B^l(p,\epsilon) := \{u \in X : \max_{i=0,\dots,l-1} \rho(\phi^i(p),\phi^i(u)) \le \epsilon\}\,.$$

Denote by $N_\epsilon(K,l)$ the smallest cardinality of a set of points $\{p_i\}_{i\in I}$ s.t. $K \subset \bigcup_{i\in I} B^l(p_i,\epsilon)$. The topological entropy of ϕ on K is defined by

$$h_{\text{top}}(\phi,K) := \lim_{\epsilon\to 0}\limsup_{l\to\infty}\frac{\log N_\epsilon(K,l)}{l}\,. \tag{1}$$

According to R. Bowen [3] we define the topological entropy of ϕ by

$$h_{\text{top}}(\phi) := \sup_{K\subset M,\,K \text{ compact}} h_{\text{top}}(\phi,K) \tag{2}$$

Fundamental properties

(1) $h_{\text{top}}(\phi,K) \ge 0$
(2) $h_{\text{top}}(\phi,K) \le h_{\text{top}}(\phi,K')$ if $K \subset K'$
(3) $h^\rho_{\text{top}}(\phi,K) = h^{\widetilde\rho}_{\text{top}}(\phi,K)$ if ρ and $\widetilde\rho$ are uniformly equivalent metrics.
(4) $(X,\rho), (\widetilde X,\widetilde\rho)$, $\widetilde\phi : \widetilde X \to \widetilde X$ continuous,
$\psi : X \to \widetilde X$ uniformly continuous map s.t. $\psi\circ\phi = \widetilde\phi\circ\psi$. Then

$$h_{\text{top}}(\widetilde\phi,\psi(K)) \le h_{\text{top}}(\phi,K)$$

If ψ^{-1} uniformly continuous (i.e. ϕ and $\widetilde\phi$ topologically conjugate) then

$$h_{\text{top}}(\widetilde\phi,\psi(K)) = h_{\text{top}}(\phi,K).$$

(5) $h_{\text{top}}(\phi^l,K) \le l\cdot h_{\text{top}}(\phi,K)$ for any $l\in\mathbb{N}$.
$h_{\text{top}}(\phi^l,K) = l\cdot h_{\text{top}}(\phi,K)$ if ϕ is uniformly continuous.

Volume growth

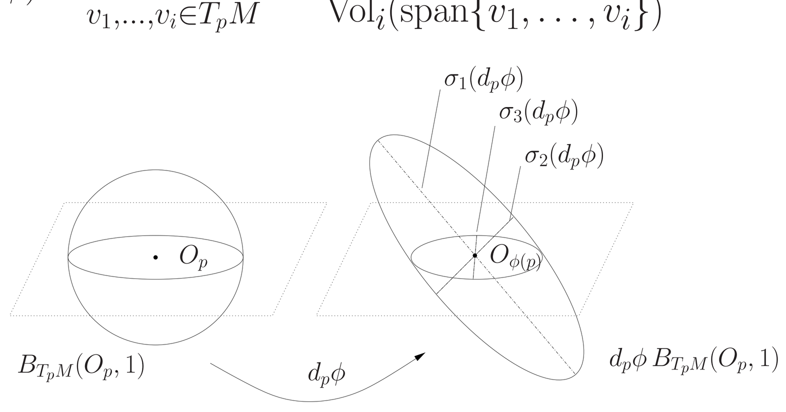
Let (M,g) be an m-dimensional Riemannian C^3-manifold, let $\phi : M\to M$ be a C^1-smooth map for which $\inf_{p\in M}|\det d_p\phi| > 0$ is satisfied.

Introduce the singular values $\sigma_1(d_p\phi) \ge \ldots \ge \sigma_m(d_p\phi) \ge 0$ of the tangent map $d_p\phi : (T_pM, \langle\cdot,\cdot\rangle_{T_pM}) \to (T_{\phi(p)}M, \langle\cdot,\cdot\rangle_{T_{\phi(p)}M})$ which by definition are the eigenvalues of self-adjoint, positive semi-definite linear operator $((d_p\phi)^*d_p\phi)^{1/2}$.
For $i\in(0,m]$ we define the inverse singular value function of $d_p\phi$ of order i

$$\omega_i(d_p\phi) := \sigma_m(d_p\phi)\cdot\ldots\cdot\sigma_{m-\lfloor i\rfloor+1}(d_p\phi)\sigma_{m-\lfloor i\rfloor}(d_p\phi)^{i-\lfloor i\rfloor}\,.$$

Here $\lfloor i\rfloor$ denotes the largest integer less than i.
The volume-growth of parallel-pipeds spanned by i vectors $v_i\in T_pM$ under the action of ϕ in first approximation is mediated by $\omega_i(d_p\phi)$: for $i\in\{1,\dots,m\}$

$$\omega_i(d_p\phi) = \inf_{v_1,\dots,v_i\in T_pM}\frac{\mathrm{Vol}_i(\mathrm{span}\{d_p\phi v_1,\dots,d_p\phi v_i\})}{\mathrm{Vol}_i(\mathrm{span}\{v_1,\dots,v_i\})}$$



We call ϕ volume-expanding on K with respect to an n-dimensional sub-bundle $E\subset T_KM$ if for any $p\in K$ and any $t>0$

$$\omega_n(d_p\phi^t{}_{|E_p}) = |\det d_p\phi^t{}_{|E_p}| > 1\,.$$

Note that

$$\sup_{E_p\subset T_pM}|\det d_p\phi^t{}_{|E_p}| \equiv \|(d_p\phi^t)^\wedge\| = \prod_{\sigma_i(d_p\phi^t)\ge 1}\sigma_i(d_p\phi^t)\,.$$

Volume growth relates dimension and entropy

The box dimension $\overline{\dim}_{\text{B}}K$ resp. the Hausdorff dimension $\dim_{\text{H}}K$ of a compact set K and the topological entropy $h_{\text{top}}(\phi,K)$ are closely related. We have for instance [1]:

Let $K\subset M$ be compact and ϕ-invariant and let $E\subset T_KM$ be equivariant w.r.t. $\varphi^{(\cdot)}(\cdot)$, e.g. $d_p\phi E_p = E_{\phi(p)}$ for all $p\in K$. Then

$$\dim_{\text{H}}K \le \overline{\dim}_{\text{B}}K \le \mathrm{codim}E + \inf\left\{D : h_{\text{top}}(\phi,K) < \inf_{p\in K}\log\omega_D(d_p\phi_{|E_p})\right\}\,. \tag{3}$$

Lower bounds of the topological entropy

Let $K\subset TM$ be a compact set and let $E\subset T_KM$ be a subbundle.
Denote $D := \dim_{\text{H}}K - \mathrm{codim}E$.
Suppose that $\log\sigma_{n-\lfloor D\rfloor}(d_p\varphi^t)/t$ is uniformly bounded for all $t>0$ and all $p\in K$

Relations between dimension and topological entropy (cf. [1],[2]) can be formulated as lower bounds of the topological entropy. First, we consider the

General case: $K\subset M$ not necessarily invariant w.r.t. ϕ.

The minimal expansion rate of $(\dim_{\text{H}}K)$-volumia minorizes the entropy:

$$h_{\text{top}}(\phi,K) \ge \left(\liminf_{t\to\infty}\frac{1}{t}\log\inf_{p\in K}\omega_{\dim_{\text{H}}K}(d_p\phi^t)\right)_+ \tag{4}$$

Moreover, taking into account a subbundle $E\subset T_KM$ we have:

$$h_{\text{top}}(\phi,K) \ge \left(\liminf_{t\to\infty}\frac{1}{t}\log\inf_{p\in K}\omega_D(d_p\phi^t{}_{|E_p})\right)_+ \tag{5}$$

Further we have:

If K is a set of full dimension $\dim_{\text{H}}K = m$ then the estimate (4) can be strengthened to

$$h_{\text{top}}(\phi,K) \ge \left(\liminf_{t\to\infty}\frac{1}{t}\log\inf_{p\in K}\|(d_p\phi^t)^\wedge\|\right)_+\,. \tag{6}$$

Compact case: $K\subset M$ invariant w.r.t. ϕ.

Suppose that K is compact and ϕ-invariant and that $E\subset T_KM$ is equivariant w.r.t. $\varphi^{(\cdot)}(\cdot)$.

In this case we have the following subexponential behavior

$$\inf_{p\in K}\omega_D(d_p\phi^t{}_{|E_p}) \ge \left(\inf_{p\in K}\omega_D(d_p\phi_{|E_p})\right)^t\,.$$

Thus, (5) yields

$$h_{\text{top}}(\phi,K) \ge \left(\sup_{t\ge 0}\frac{1}{t}\log\inf_{p\in K}\omega_D(d_p\phi^t{}_{|E_p})\right)_+ \tag{7}$$

Making use of the global inverse Lyapunov exponents of ϕ on K w.r.t. E which are recursively defined by $(i=1,\dots,n)$

$$\nu_n^E + \ldots + \nu_{n-i+1}^E := \lim_{t\to\infty}\frac{1}{t}\log\inf_{p\in K}\omega_i(d_p\phi^t{}_{|E_p})$$

we obtain

$$h_{\text{top}}(\varphi^1,K) \ge \left(\nu_n^E + \ldots + \nu_{n-\lfloor D\rfloor+1}^E + (D-\lfloor D\rfloor)\nu_{n-\lfloor D\rfloor}^E\right)_+ \tag{8}$$

Global inverse Lyapunov exponents are closely related to the commonly used measure-theoretic Lyapunov exponents:
Denote by $\mathcal{E}(K,\phi)$ the set of invariant ergodic probability measures supported on K. For any $\mu\in\mathcal{E}(K,\phi)$ there exist numbers $\nu_1(\mu)\ge\ldots\ge\nu_m(\mu)$ (the Lyapunov exponents w.r.t. μ) such that $\log\sigma_i(d_p\phi^t)/t \to \nu_i(\mu)$ for μ-a.e. point $p\in K$ as $t\to\infty$. For $E = T_KM$ there holds

$$\sum_{i=n}^{n-D}\nu_i^E = \inf_{\mu\in\mathcal{E}(K,\phi)}\sum_{i=n}^{n-D}\nu_i(\mu)\,. \tag{9}$$

Examples

We focus especially on time-discrete and time-continuous dynamical systems (generated by maps resp. flows) with non-compact phase space.

Linear maps in $\mathbb{R}^m$ Let $L\in M_m(\mathbb{R})$ be a matrix with eigenvalues $\lambda_1,\dots,\lambda_m$. Consider the map

$$\phi : \mathbb{R}^m \to \mathbb{R}^m : (x_1,\dots,x_m)^T \mapsto L(x_1,\dots,x_m)^T$$

and the $d\phi$-invariant subbundle $E\subset T\mathbb{R}^m = \mathbb{R}^m\times\mathbb{R}^m$ which corresponds to the eigenspace of the eigenvalues λ_i with $|\lambda_i|>1$.
Note that $\log|\det(d_p\phi_{|E_p})| = \sum_{|\lambda_i|>1}\log|\lambda_i|$ for any $p\in\mathbb{R}^m$.
Thus, for any compact set $K\subset\mathbb{R}^3$ of full Hausdorff dimension $\dim_{\text{H}}K = m$ we have by definition (2) and by (4)

$$h_{\text{top}}(\phi) \ge h_{\text{top}}(\phi,K) \ge \sum_{|\lambda_i|>1}\log|\lambda_i|\,.$$

This estimate is sharp (cf. [3]).

Hénon maps Consider the family of polynomial diffeomorphisms given by

$$\phi : \mathbb{R}^2\to\mathbb{R}^2 : (x,y)^T \mapsto (p(x)+by, x)$$

p real polynomial of degree two, $b\in\mathbb{R}$. Eq. (4) and definition (2) imply

$$h_{\text{top}}(\phi) \ge (\log|b|)_+\,.$$

Lorenz system Consider the global flow $\varphi^{(\cdot)}(\cdot) : \mathbb{R}\times\mathbb{R}^3\to\mathbb{R}^3$ for the Lorenz system $(b=8/3, \sigma=10, r=28)$

$$\begin{aligned}\dot x &= -\sigma x + \sigma y\\ \dot y &= rx - y - xz\\ \dot z &= -bz + xy\end{aligned}$$

Then, by definition (2) and (4)

$$h_{\text{top}}(\varphi^{-1}) \ge \sigma + 1 + b = \frac{41}{3}\,.$$

Geodesic flow Let (M,g) be an m-dimensional complete Riemannian C^4-smooth manifold. Consider the geodesic flow $\varphi^{(\cdot)}(\cdot) : \mathbb{R}\times SM\to SM$ on the unit tangent bundle SM. Let $K\subset SM$ be compact and of full Hausdorff dimension $2m-1$.

(A1) Suppose that (M,g) has no conjugate points and the sectional curvature is uniformly bounded from below, i.e. $\exists\,k_1>0$ s.t. $\forall p\in M, \forall v,w\in T_pM$ with $||v||_{T_pM} = ||w||_{T_pM} = 1$

$$-k_1 \le R_p(v,w)w\,.$$

Here R denotes the Riemannian curvature tensor.
Then we obtain by (4)

$$h_{\text{top}}(\varphi^1,K) \ge \left(\liminf_{t\to\infty}\frac{1}{t}\log\inf_{\theta\in K}\inf_{\widetilde\theta=(\widetilde p,\widetilde v)\in\mathcal{O}_+(t,\theta)}\sqrt{|\mathrm{tr}\,(R_{\widetilde p}(\cdot,\widetilde v)\widetilde v)|}\right)_+\,,$$

where $\mathcal{O}_+(t,\theta)$ denotes the semi-orbit of length t trough θ.

(A2) Suppose that M is compact and that M has negative sectional curvature. i.e. $\exists\,k_2>0$ s.t. $\forall p\in M$ and $\forall v,w\in T_pM$

$$\langle R_p(v,w)w,v\rangle < -k_2\,.$$

In this case $\varphi^{(\cdot)}(\cdot)$ is an Anosov flow. Using the global inverse Lyapunov exponents w.r.t. the instable subbundle E inequality (8) implies the following estimate which is sharp in the case of constant sectional curvature

$$h_{\text{top}}(\varphi,K) \ge \nu_1^E + \ldots + \nu_{m-1}^E \ge (m-1)\inf_{(p,v),(p,w)\in SM}\sqrt{|\langle R_p(v,w)w,v\rangle|}\,.$$

References

[1] K. Gelfert, Estimates of the box dimension and of the topological entropy for volume-contracting and partially volume-expanding dynamical systems on manifolds. Dissertation TU Dresden (2001).
[2] K. Gelfert, Lower bounds of the topological entropy. Submitted to *Dynamical Systems* (2002).
[3] R. Bowen, Entropy for group endomorphism and homogeneous spaces, Trans. Am. Math. Soc. **153** (1971).

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